Lucas-Kanade Tracker

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Simple Kanade-Lucas-Tomasi (KLT) Algorithm

1. Detect Harris corners in the first frame

2. For each Harris corner, compute motion (translation or affine) between consecutive frames

3. Link motion vectors in successive frames to get a track for each Harris point

4. Introduce new Harris points by applying Harris detector at every m (10 or 15) frames

5. Track new and old Harris points using steps 1-3

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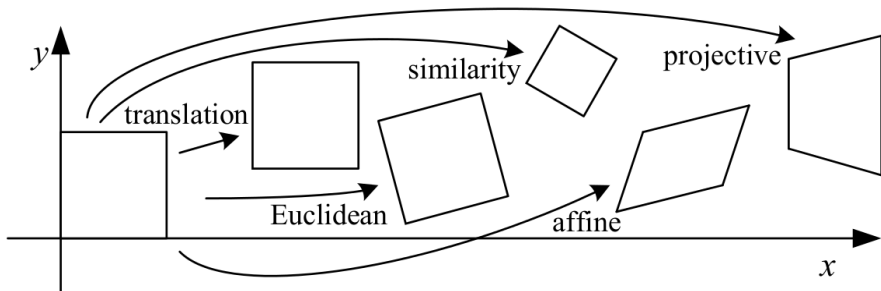5. Track new and old Harris points using steps 1-3

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- Need to register a patch of the current frame to another patch of the next frame
- Coordinate transformation can be done by different "motions"

Summary of displacement models (2-D transformations)

- Translation:
  $$x' = x + b_1$$
  $$y' = y + b_2$$

- Rigid:
  $$x' = x \cos \theta - y \sin \theta + b_1$$
  $$y' = x \sin \theta + y \cos \theta + b_2$$

- Affine:
  $$x' = a_1 x + a_2 y + b_1$$
  $$y' = a_3 x + a_4 y + b_2$$

- Projective:
  $$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$
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Approximate transformations

- Bi-quadratic:
  $$x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$$
  $$y' = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} y^2 + a_{12} xy$$

- Bi-linear:
  $$x' = a_1 + a_2 x + a_3 y + a_4 xy$$
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- Pseudo-perspective:
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Review of Taylor series expansion

Consider first order approximation of a scalar function $f(x)$, from undergrad calculus,

$$f(x_0 + \Delta x) \approx f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} \Delta x$$

Now consider a vector function $F(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_M(\mathbf{x})]^T$, where $\mathbf{x} = [x_1, x_2, \cdots, x_N]^T$, we have

$$f_1(\mathbf{x}_0 + \Delta \mathbf{x}) \approx f_1(\mathbf{x}_0) + \left.\frac{\partial f_1(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_1 + \cdots + \left.\frac{\partial f_1(\mathbf{x})}{\partial x_N}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_N$$

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So we have,

$$F(\mathbf{x}_0 + \Delta\mathbf{x}) \approx F(\mathbf{x}_0) + \underbrace{\left.\begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1}, \frac{\partial f_1(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_1(\mathbf{x})}{\partial x_N} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1}, \frac{\partial f_2(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_2(\mathbf{x})}{\partial x_N} \\ \cdots \\ \frac{\partial f_M(\mathbf{x})}{\partial x_1}, \frac{\partial f_M(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_M(\mathbf{x})}{\partial x_N} \end{pmatrix}\right|_{\mathbf{x}=\mathbf{x}_0}}_{\frac{\partial F(\mathbf{x}_0)}{\partial \mathbf{x}}} \Delta\mathbf{x},$$

where we denote the matrix as $\frac{\partial F(\mathbf{x}_0)}{\partial \mathbf{x}}$, which is also known to be the Jacobian of $F(\cdot)$ w.r.t $\mathbf{x}$ at point $\mathbf{x}_0$

Finding alignment

- Goal: Given template $T(\mathbf{x})$, find $\mathbf{p}$ to minimize

$$\sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x})]^2$$

- Consider $\mathbf{p}_0 + \Delta \mathbf{p}$, $\mathbf{p}_0$ is optimum if

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x}; \mathbf{p}_0 + \Delta \mathbf{p})) - T(\mathbf{x})]^2 = 0$$

- By Taylor series expansion,

$$\sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p}_0 + \Delta \mathbf{p})) - T(\mathbf{x})]^2$$

$$\approx \sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \Delta \mathbf{p} - T(\mathbf{x})]^2$$

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$$\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T [T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p}_0))],$$

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$$= 2 \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x}; \mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \Delta \mathbf{p} - T(\mathbf{x})] = 0$$

$$\Rightarrow \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right] \Delta \mathbf{p} =$$

$$\sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T [T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p}_0))]$$

$$\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T [T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p}_0))],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]^T \left[(\nabla I)^T \frac{\partial W(\mathbf{x}; \mathbf{p}_0)}{\partial \mathbf{p}} \right]$

Example: Hessian for translation motion

For translation motion, we may write $W(\mathbf{x}; \mathbf{p}) = \mathbf{x} + \mathbf{p}$, thus

$$\frac{\partial W}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial}{\partial p_1}(x_1 + p_1) & \frac{\partial}{\partial p_2}(x_1 + p_1) \\ \frac{\partial}{\partial p_1}(x_2 + p_2) & \frac{\partial}{\partial p_2}(x_2 + p_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \text{ Then}$$

$$(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{pmatrix}$$

and

$$H = \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p_0}} \right]^T \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p_0}} \right] = \sum_{\mathbf{x}} \begin{pmatrix} \left(\frac{\partial I}{\partial x}\right)^2 & \frac{\partial I}{\partial x} \frac{\partial I}{\partial y} \\ \frac{\partial I}{\partial x} \frac{\partial I}{\partial y} & \left(\frac{\partial I}{\partial y}\right)^2 \end{pmatrix},$$

which btw is really a Harris corner detector

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Computing the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$

Richard Szeliski, "Computer Vision: Algorithms and Applications"

Transformation	Matrix	# DoF	Preserves	Icon	Parameters p	Jacobian J
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation		(t_x, t_y)	$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths		(t_x, t_y, θ)	$\left[\begin{array}{cccc} 1 & 0 & -s_\theta x - c_\theta y \\ 0 & 1 & c_\theta x - s_\theta y \end{array}\right]$
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles		(t_x, t_y, a, b)	$\left[\begin{array}{cccc} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{array}\right]$
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism		$(t_x, t_y, a_{00}, a_{01}, a_{10}, a_{11})$	$\left[\begin{array}{cccccc} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{array}\right]$
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]_{3\times 3}$	8	straight lines		$(h_{00}, h_{01}, \ldots, h_{21})$	(see Section 6.1.3)

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$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p})) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract I from T
3. Compute gradient ∇I
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{p})$
5. Compute the steepest descent $(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}}$
6. Compute inverse Hessian $H = \sum_{\mathbf{x}} \left((\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right)$
7. Compute descend-error product $\sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p})) \right)$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

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Some variations of Lucas-Kanade algorithms

- Instead of considering

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p_0} + \Delta \mathbf{p})) - T(\mathbf{x})]^2 = 0$$

- We can approximate the above as

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(W(\mathbf{x}; \mathbf{p_0}); \Delta \mathbf{p}))) - T(\mathbf{x})]^2 = 0$$

  If we go through the same deviation, this will lead to the so-called "compositional algorithm"

- More interestingly, note that our goal is that $\mathbf{p}_0$ should be stationary w.r.t. any $\Delta \mathbf{p}$, therefore we can also consider ("inverse compositional alignment")

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(W(\mathbf{x}; \mathbf{p}_0); -\Delta \mathbf{p})) - T(\mathbf{x})]^2$$

$$\approx \frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(\mathbf{x}; \mathbf{p}_0)) - T(W(\mathbf{x}; \Delta \mathbf{p}))]^2 = 0$$

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$$\approx \frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p}_0)) - T(W(\mathbf{x}; \mathbf{0})) - (\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}]^2$$

$$= -2 \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x}; \mathbf{p}_0)) - T(\mathbf{x}) - (\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}] = 0$$

$$\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x}; \mathbf{p}_0)) - T(\mathbf{x})],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]$

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$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x}; \mathbf{p}_0))) - T(W(\mathbf{x}; \Delta \mathbf{p}))]^2 = 0$$

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$$\approx \frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p}_0)) - T(W(\mathbf{x}; \mathbf{0})) - (\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}]^2$$

$$= -2 \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x}; \mathbf{p}_0)) - T(\mathbf{x}) - (\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}] = 0$$

$$\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x}; \mathbf{p}_0)) - T(\mathbf{x})],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]^T \left[(\nabla T)^T \frac{\partial W(\mathbf{x}; \mathbf{0})}{\partial \mathbf{p}} \right]$

$$\Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x};\mathbf{p})$

2. Subtract T from I

3. Compute gradient ∇T (only do once)

4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x};\mathbf{0})$ (only do once)

5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)

6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)

7. Multiply steepest descend with error
   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$

8. Compute $\Delta\mathbf{p}$

9. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta\mathbf{p}$

(Inverse compositional) Modified Lucas-Kanade
Baker *et al.*, IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
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4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)
7. Multiply steepest descend with error
   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$
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(Inverse compositional) Modified Lucas-Kanade

$$\Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$$

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(Inverse compositional) Modified Lucas-Kanade
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$$\Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
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7. Multiply steepest descend with error $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

$$\Delta\mathbf{p} = H^{-1}\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial\mathbf{p}}\right]^T\left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right)$$

1. Warp I with $W(\mathbf{x};\mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial\mathbf{p}}$ at $(\mathbf{x};\mathbf{0})$ (only do once)
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7. Multiply steepest descend with error $\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial\mathbf{p}}\right]^T\left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right)$
8. Compute $\Delta\mathbf{p}$
9. Update parameters $\mathbf{p} \to \mathbf{p} + \Delta\mathbf{p}$

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)
7. Multiply steepest descend with error
   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \to \mathbf{p} + \Delta \mathbf{p}$

(Inverse compositional) Modified Lucas-Kanade
Baker *et al.*, IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)
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   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \to \mathbf{p} + \Delta \mathbf{p}$

(Inverse compositional) Modified Lucas-Kanade
Baker *et al.*, IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)
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   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

(Inverse compositional) Modified Lucas-Kanade

Baker *et al.*, IJCV 2004

$$\Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x};\mathbf{p})$
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5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
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   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$
8. Compute $\Delta\mathbf{p}$
9. Update parameters $\mathbf{p} \to \mathbf{p} + \Delta\mathbf{p}$

(Inverse compositional) Modified-KLT
Baker *et al.*, IJCV 2004

Initialize:

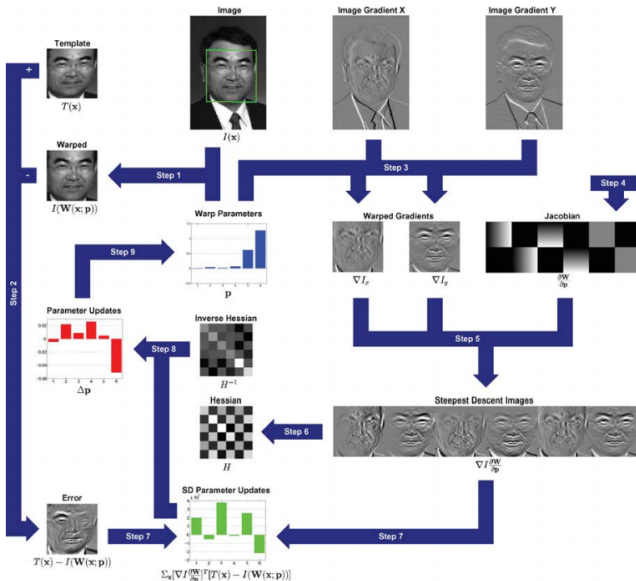1. Compute gradient ∇T

2. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$

3. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$

4. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$

Loop:

1. Warp I with $W(\mathbf{x}; \mathbf{p})$

2. Subtract T from I

3. Multiply steepest descend with error
   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$

4. Compute $\Delta \mathbf{p}$

5. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

Modified Lucas-Kanade
Baker *et al.*, IJCV 2004

References

- Simon Baker and Iain Matthews, "Lucas-Kanade 20 Years On: A Unifying Framework," IJCV, 2004
- Section 8.2, Richard Szeliski, "Computer Vision: Algorithms and Applications"

Implementations

- OpenCV implementation: http://www.ces.clemson.edu/~stb/klt/
- Some Matlab Implementation: Lucas Kanade with Pyramid
  - http://www.mathworks.com/matlabcentral/fileexchange/30822
  - Affine tracking: http://www.mathworks.com/matlabcentral/fileexchange/24677-lucas-kanade-affine-template-tracking
  - http://vision.eecs.ucf.edu/Code/Optical_Flow/Lucas%20Kanade.zip

Some follow-up works from UCF

- Hamid Izadinia, Imran Saleemi, Wenhui Li, and Mubarak Shah, "$(MP)^2T$: Multiple People Multiple Parts Tracker," ECCV 2012
- Amir Roshan Zamir, Afshi Dehghan, and Mubarak Shah, "GMCP-Tracker: Global Multi-object Tracking Using Generalized Minimum clique Graphs," ECCV 2012