ECE 4973: Lecture 12
Kanade-Lucas-Tomasi (KLT) Tracker

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Slides inspired by Prof Shah's lecture at UCF

Simple Kanade-Lucas-Tomasi (KLT) Algorithm

1. Detect Harris corners in the first frame
2. For each Harris corner, compute motion (translation or affine) between consecutive frames
3. Link motion vectors in successive frames to get a track for each Harris point
4. Introduce new Harris points by applying Harris detector at every m (10 or 15) frames
5. Track new and old Harris points using steps 1-3

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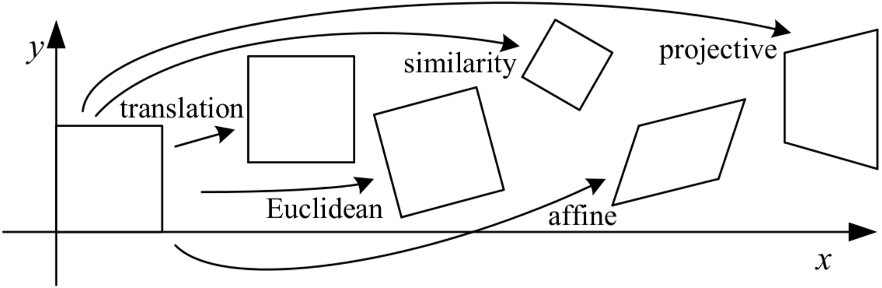
- Need to register a patch of the current frame to another patch of the next frame
- Coordinate transformation can be done by different "motions"



Summary of displacement models (2-D transformations)

- Translation:
  $x' = x + b_1$
  $y' = y + b_2$
- Rigid:
  $x' = x\cos\theta - y\sin\theta + b_1$
  $y' = x\sin\theta + y\cos\theta + b_2$
- Affine:
  $x' = a_1 x + a_2 y + b_1$
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- Projective:
  $x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$
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Approximate transformations

- Bi-quadratic:
  $x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$
  $y' = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} y^2 + a_{12} xy$
- Bi-linear:
  $x' = a_1 + a_2 x + a_3 y + a_4 xy$
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- Pseudo-perspective:
  $x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$
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Review of Taylor series expansion

Consider first order approximation of a scalar function $f(x)$, from undergrad calculus,

$$f(x_0 + \Delta x) \approx f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} \Delta x$$

Now consider a vector function $F(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_M(\mathbf{x})]^T$, where $\mathbf{x} = [x_1, x_2, \cdots, x_N]^T$, we have

$$f_1(\mathbf{x}_0 + \Delta\mathbf{x}) \approx f_1(\mathbf{x}_0) + \left.\frac{\partial f_1(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_1 + \cdots + \left.\frac{\partial f_1(\mathbf{x})}{\partial x_N}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_N$$

$$\cdots$$

$$f_M(\mathbf{x}_0 + \Delta\mathbf{x}) \approx f_M(\mathbf{x}_0) + \left.\frac{\partial f_M(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_1 + \cdots + \left.\frac{\partial f_M(\mathbf{x})}{\partial x_N}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta x_N,$$

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Review of Jacobian

So we have,

$$F(\mathbf{x}_0 + \Delta\mathbf{x}) \approx F(\mathbf{x}_0) + \underbrace{\left.\begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1}, \frac{\partial f_1(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_1(\mathbf{x})}{\partial x_N} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1}, \frac{\partial f_2(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_2(\mathbf{x})}{\partial x_N} \\ \cdots \\ \frac{\partial f_M(\mathbf{x})}{\partial x_1}, \frac{\partial f_M(\mathbf{x})}{\partial x_2}, \cdots, \frac{\partial f_M(\mathbf{x})}{\partial x_N} \end{pmatrix}\right|_{\mathbf{x}=\mathbf{x}_0}}_{\frac{\partial F(\mathbf{x}_0)}{\partial \mathbf{x}}} \Delta\mathbf{x},$$

where we denote the matrix as $\frac{\partial F(\mathbf{x}_0)}{\partial \mathbf{x}}$, which is also known to be the Jacobian of $F(\cdot)$ w.r.t $\mathbf{x}$ at point $\mathbf{x}_0$

Finding alignment

- Goal: Given template $T(\mathbf{x})$, find $\mathbf{p}$ to minimize

$$\sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})]^2$$

- Consider $\mathbf{p}_0 + \Delta\mathbf{p}$, $\mathbf{p}_0$ is optimum if

$$\frac{\partial}{\partial \Delta\mathbf{p}} \sum_{x} [I(W(\mathbf{x};\mathbf{p}_0 + \Delta\mathbf{p})) - T(\mathbf{x})]^2 = 0$$

- By Taylor series expansion,

$$\sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0 + \Delta\mathbf{p})) - T(\mathbf{x})]^2$$

$$\approx \sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x})]^2$$

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$$=2\sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T [I(W(\mathbf{x};\mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x})] = 0$$

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$$\therefore \Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T [T(\mathbf{x}) - I(W(\mathbf{x};\mathbf{p}_0))],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]$

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$$\frac{\partial}{\partial \boldsymbol{\Delta}\mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x})]^2$$

$$=2\sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T [I(W(\mathbf{x};\mathbf{p}_0)) + (\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x})] = 0$$

$$\Rightarrow \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right] \Delta\mathbf{p} =$$

$$\sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T [T(\mathbf{x}) - I(W(\mathbf{x};\mathbf{p}_0))]$$

$$\therefore \Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W(\mathbf{x};\mathbf{p}_0)}{\partial \mathbf{p}}\right]^T [T(\mathbf{x}) - I(W(\mathbf{x};\mathbf{p}_0))],$$

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Example: Hessian for translation motion

For translation motion, we may write $W(\mathbf{x};\mathbf{p}) = \mathbf{x} + \mathbf{p}$, thus $\frac{\partial W}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial}{\partial p_1}(x_1+p_1) & \frac{\partial}{\partial p_2}(x_1+p_1) \\ \frac{\partial}{\partial p_1}(x_2+p_2) & \frac{\partial}{\partial p_2}(x_2+p_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{pmatrix}$$

and

$$H = \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}_0} \right]^T \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}_0} \right] = \sum_{\mathbf{x}} \begin{pmatrix} \left(\frac{\partial I}{\partial x}\right)^2 & \frac{\partial I}{\partial x}\frac{\partial I}{\partial y} \\ \frac{\partial I}{\partial x}\frac{\partial I}{\partial y} & \left(\frac{\partial I}{\partial y}\right)^2 \end{pmatrix},$$

which btw is the same matrix we saw in a Harris corner detector

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Computing the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$

Richard Szeliski, "Computer Vision: Algorithms and Applications"

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

Parameters p	Jacobian J
(t_x, t_y)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(t_x, t_y, θ)	$\begin{bmatrix} 1 & 0 & -s_\theta x - c_\theta y \\ 0 & 1 & c_\theta x - s_\theta y \end{bmatrix}$
(t_x, t_y, a, b)	$\begin{bmatrix} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{bmatrix}$
$(t_x, t_y, a_{00}, a_{01}, a_{10}, a_{11})$	$\begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{bmatrix}$
$(h_{00}, h_{01}, \ldots, h_{21})$	(see Section 6.1.3)

Kanade-Lucas-Tomasi

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p})) \right)$$

- KLT is an iterative algorithm
  - Similar to iternative Lucas-Kanade but extend to arbitrary transform
  - $\Delta \mathbf{p} \leftarrow H^{-1} \sum_{\mathbf{x}} \left[(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(T(\mathbf{x}) - I(W(\mathbf{x}; \mathbf{p})) \right)$
  - $\mathbf{p} \leftarrow \mathbf{p} + \Delta \mathbf{p}$

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2. Subtract I from T
3. Compute gradient ∇I of warped image
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{p})$
5. Compute the product $(\nabla I)^T \frac{\partial W}{\partial \mathbf{p}}$
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Some variations of Kanade-Lucas-Tomasi algorithms

- Instead of considering

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x}; \mathbf{p_0} + \Delta \mathbf{p})) - T(\mathbf{x})]^2 = 0$$

- We can approximate the above as

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(W(\mathbf{x}; \mathbf{p}_0); \Delta \mathbf{p}))) - T(\mathbf{x})]^2 = 0$$

  If we go through the same deviation, this will lead to the so-called "compositional algorithm"

- More interestingly, note that our goal is that $\mathbf{p}_0$ should be stationary w.r.t. any $\Delta \mathbf{p}$, therefore we can also consider ("inverse compositional alignment")

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(W(\mathbf{x}; \mathbf{p}_0); -\Delta \mathbf{p})) - T(\mathbf{x})]^2$$

$$\approx \frac{\partial}{\partial \Delta \mathbf{p}} \sum_{x} [I(W(\mathbf{x}; \mathbf{p}_0)) - T(W(\mathbf{x}; \Delta \mathbf{p}))]^2 = 0$$

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$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0))) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2 = 0$

$$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2$$

$$\approx \frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\mathbf{0})) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta\mathbf{p}]^2$$

$$= -2 \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x}) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta\mathbf{p}] = 0$$

$$\therefore \Delta\mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x})],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]$

$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0))) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2 = 0$

$$\begin{aligned}
&\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2 \\
\approx &\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\mathbf{0})) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}]^2 \\
= &-2 \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x}) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}] = 0
\end{aligned}$$

$$\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x})],$$

where $H = \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]$

$\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0))) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2 = 0$

$$
\begin{aligned}
&\frac{\partial}{\partial \Delta \mathbf{p}} \sum_x [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\Delta\mathbf{p}))]^2 \\
\approx &\frac{\partial}{\partial \Delta \mathbf{p}} \sum_{\mathbf{x}} [I(W(\mathbf{x};\mathbf{p}_0)) - T(W(\mathbf{x};\mathbf{0})) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}]^2 \\
= &-2 \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x}) - (\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \Delta \mathbf{p}] = 0
\end{aligned}
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\therefore \Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T [I(W(\mathbf{x};\mathbf{p}_0)) - T(\mathbf{x})],
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where $H = \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]^T \left[(\nabla T)^T \frac{\partial W(\mathbf{x};\mathbf{0})}{\partial \mathbf{p}} \right]$

(Inverse compositional) Modified Kanade-Lucas-Tomasi
Baker et al., IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T (I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}))$$

1. Warp I with $W(\mathbf{x};\mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x};\mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$ (only do once)
7. Multiply steepest descend with error $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T (I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}))$
8. Compute $\Delta \mathbf{p}$
9. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

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$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$$

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(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}) \right)$$

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(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

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(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

$$\Delta\mathbf{p} = H^{-1}\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right]^T\left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right)$$

1. Warp I with $W(\mathbf{x};\mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x};\mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}$ (only do once)
6. Compute Hessian $H = \sum_{\mathbf{x}}\left((\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right)^T\left((\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right)$ (only do once)
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   $\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right]^T\left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right)$
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(Inverse compositional) Modified Kanade-Lucas-Tomasi
Baker et al., IJCV 2004

$$\Delta\mathbf{p} = H^{-1}\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right]^T (I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x}))$$

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8. Compute $\Delta\mathbf{p}$
9. Update parameters $\mathbf{p}\to\mathbf{p}+\Delta\mathbf{p}$

(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

$$\Delta \mathbf{p} = H^{-1} \sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T \left(I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}) \right)$$

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Compute gradient ∇T (only do once)
4. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$ (only do once)
5. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$ (only do once)
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(Inverse compositional) Modified Kanade-Lucas-Tomasi

Baker et al., IJCV 2004

$$\Delta\mathbf{p} = H^{-1}\sum_{\mathbf{x}}\left[(\nabla T)^T\frac{\partial W}{\partial \mathbf{p}}\right]^T\left(I(W(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right)$$

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2. Subtract T from I
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8. Compute $\Delta\mathbf{p}$
9. Update parameters $\mathbf{p} \to \mathbf{p} + \Delta\mathbf{p}$

(Inverse compositional) Modified-KLT
Baker et al., IJCV 2004

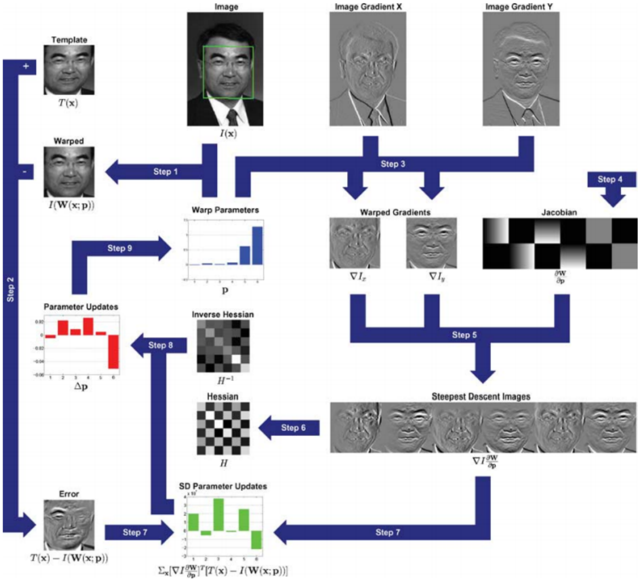
Initialize:

1. Compute gradient ∇T
2. Evaluate the Jacobian $\frac{\partial W}{\partial \mathbf{p}}$ at $(\mathbf{x}; \mathbf{0})$
3. Compute the steepest descent $(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}}$
4. Compute Hessian $H = \sum_{\mathbf{x}} \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)^T \left((\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right)$

Loop:

1. Warp I with $W(\mathbf{x}; \mathbf{p})$
2. Subtract T from I
3. Multiply steepest descend with error
   $\sum_{\mathbf{x}} \left[(\nabla T)^T \frac{\partial W}{\partial \mathbf{p}} \right]^T (I(W(\mathbf{x}; \mathbf{p})) - T(\mathbf{x}))$
4. Compute $\Delta \mathbf{p}$
5. Update parameters $\mathbf{p} \rightarrow \mathbf{p} + \Delta \mathbf{p}$

Modified Kanade-Lucas-Tomasi
Baker et al., IJCV 2004



References

- Simon Baker and Iain Matthews, "Lucas-Kanade 20 Years On: A Unifying Framework," IJCV, 2004
- Section 8.2, Richard Szeliski, "Computer Vision: Algorithms and Applications"

Implementations

- OpenCV implementation: http://www.ces.clemson.edu/~stb/klt/
- Some Matlab Implementation: Lucas Kanade with Pyramid
  - http://www.mathworks.com/matlabcentral/fileexchange/30822
  - Affine tracking: http://www.mathworks.com/matlabcentral/fileexchange/24677-lucas-kanade-affine-template-tracking
  - http://vision.eecs.ucf.edu/Code/Optical_Flow/Lucas%20Kanade.zip