Epipolar Geometry

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Slide credit: James Tompkin, Naoh Snavely

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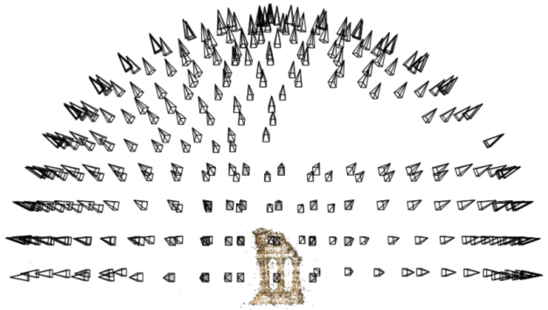
Structure from motion

Structure from motion

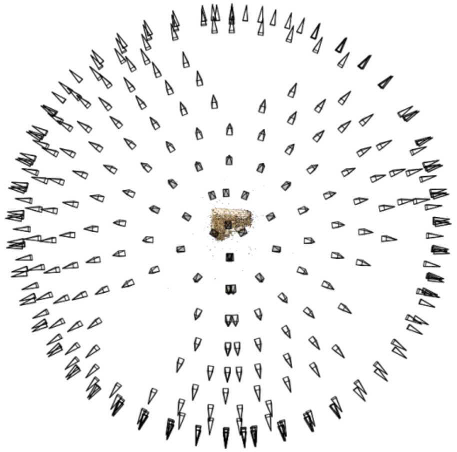
- Given many images, how can we
  a) figure out where they were all taken from?
  b) build a 3D model of the scene?

This is (roughly) the **structure from motion** problem

Structure from motion



Reconstruction (side)



(top)

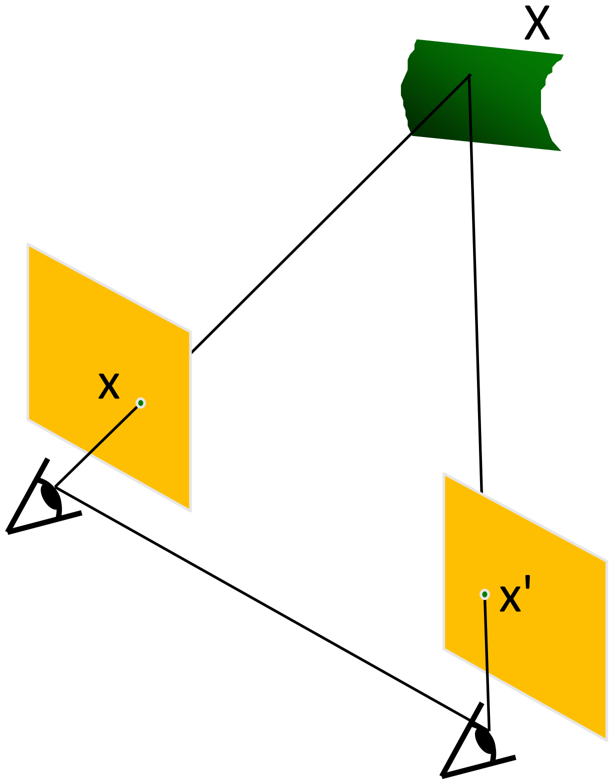
- Input: images with points in correspondence $p_{i,j} = (u_{i,j}, v_{i,j})$
- Output
  - structure: 3D location $\mathbf{x}_i$ for each point p_i
  - motion: camera parameters $\mathbf{R}_j$, $\mathbf{t}_j$ possibly $\mathbf{K}_j$
- Objective function: minimize *reprojection error*

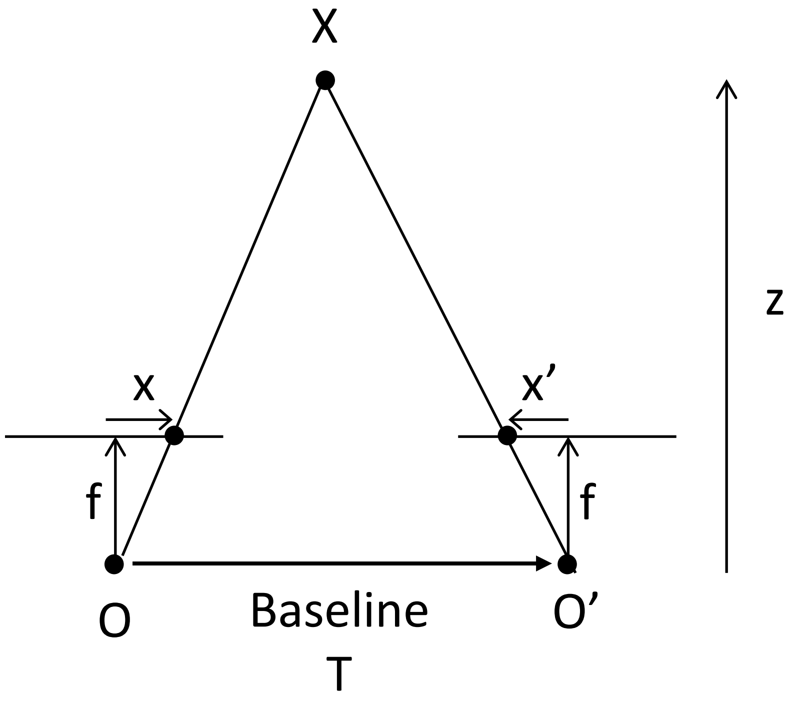
What we’ve seen so far…

- 2D transformations between images
  - Translations, affine transformations, homographies…
- 3D coordinates to 2D coordinates
  - Camera matrix
- Today: epipolar geometry and fundamental matrices

Depth from disparity

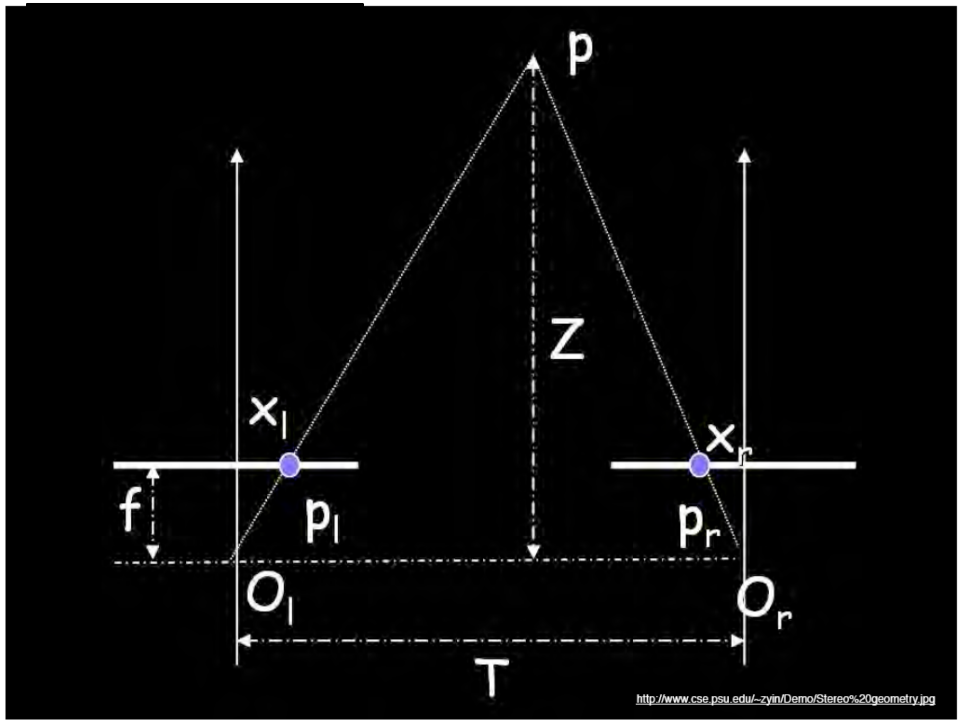
- Goal: recover depth by finding image coordinate x’ that corresponds to x





Geometry for a simple stereo system

- Assume parallel optical axes, known camera parameters (i.e., calibrated cameras). **What is expression for Z?**



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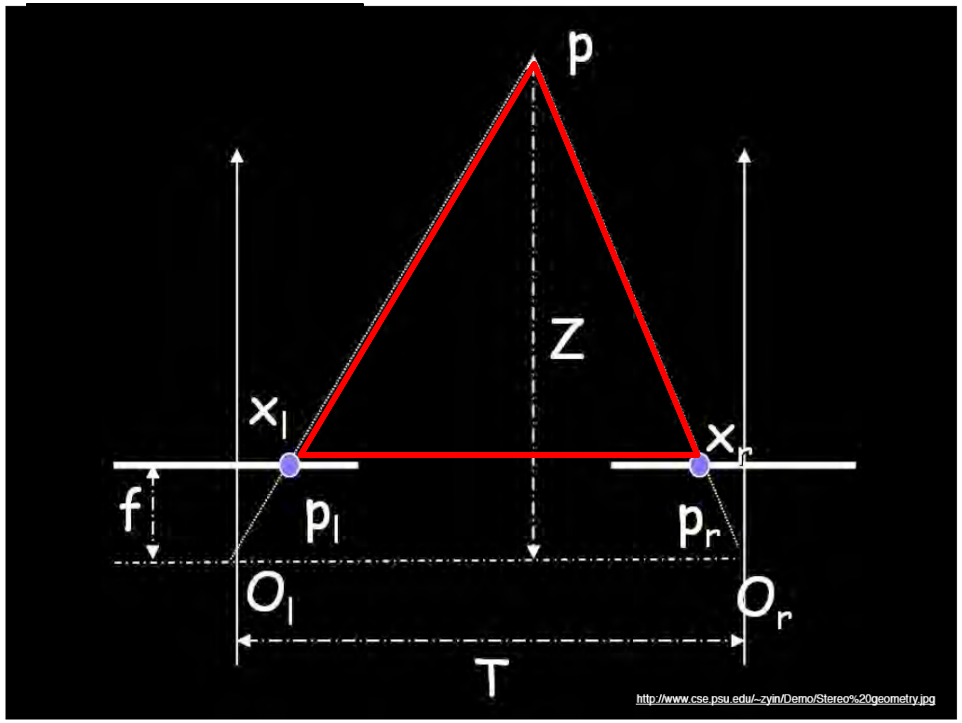

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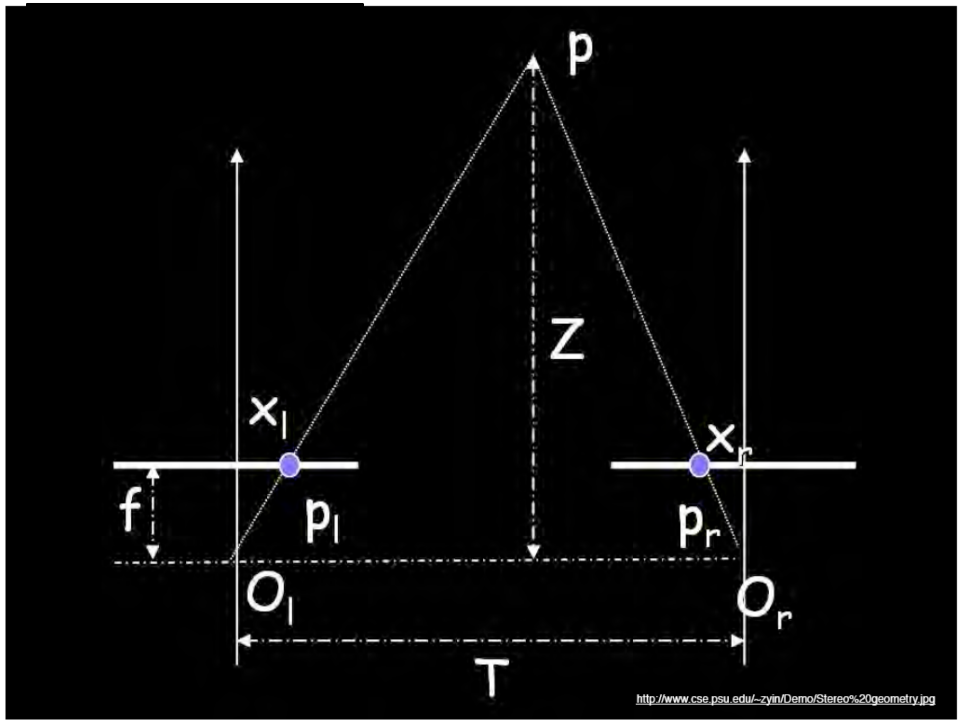

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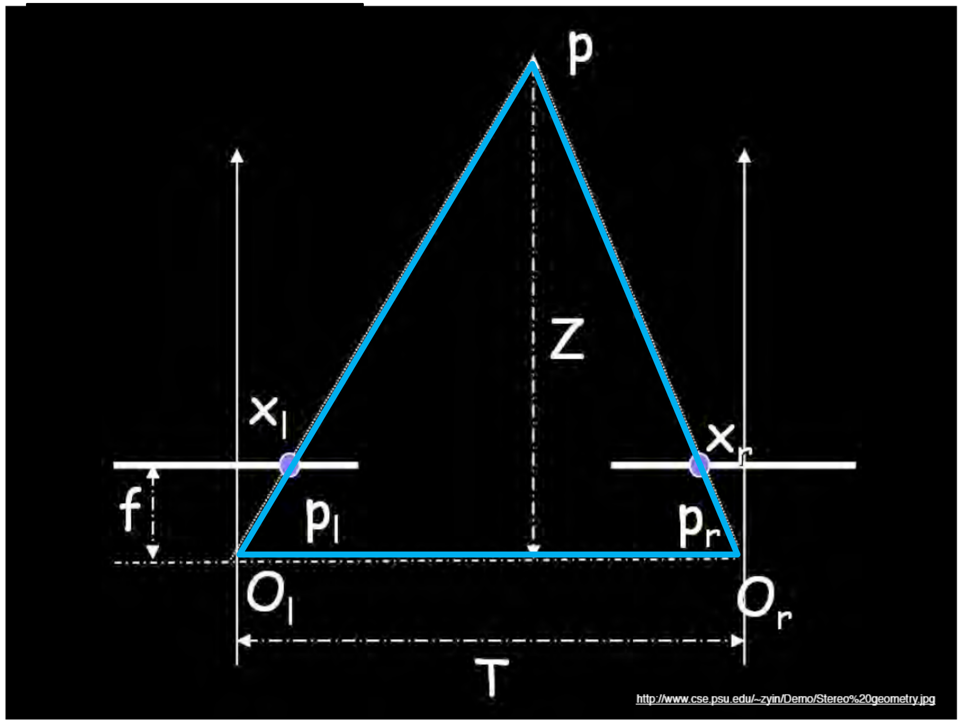

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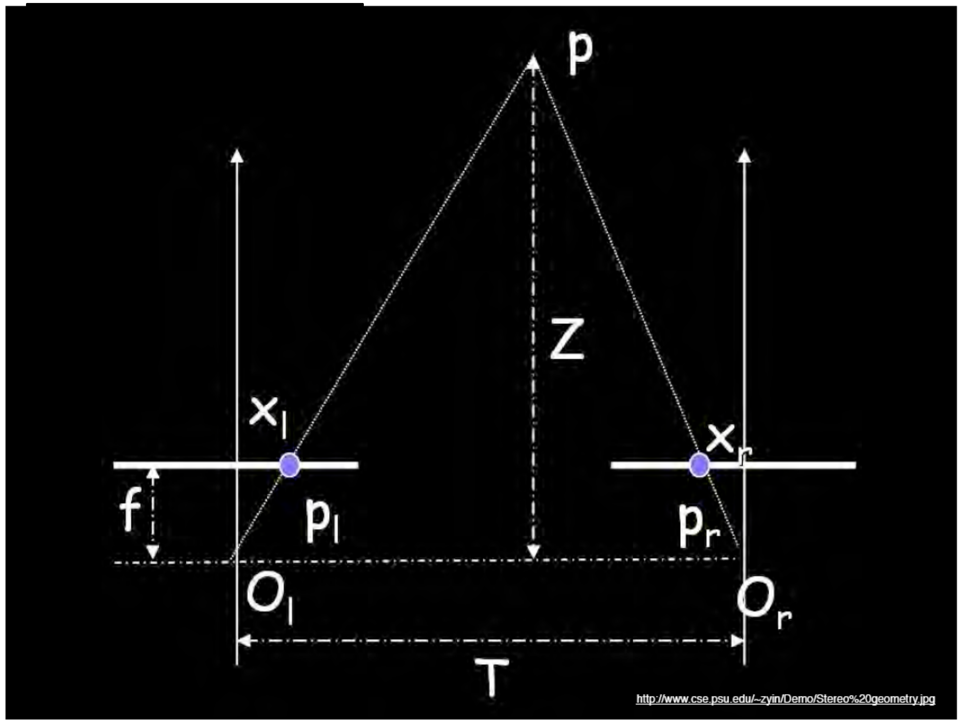

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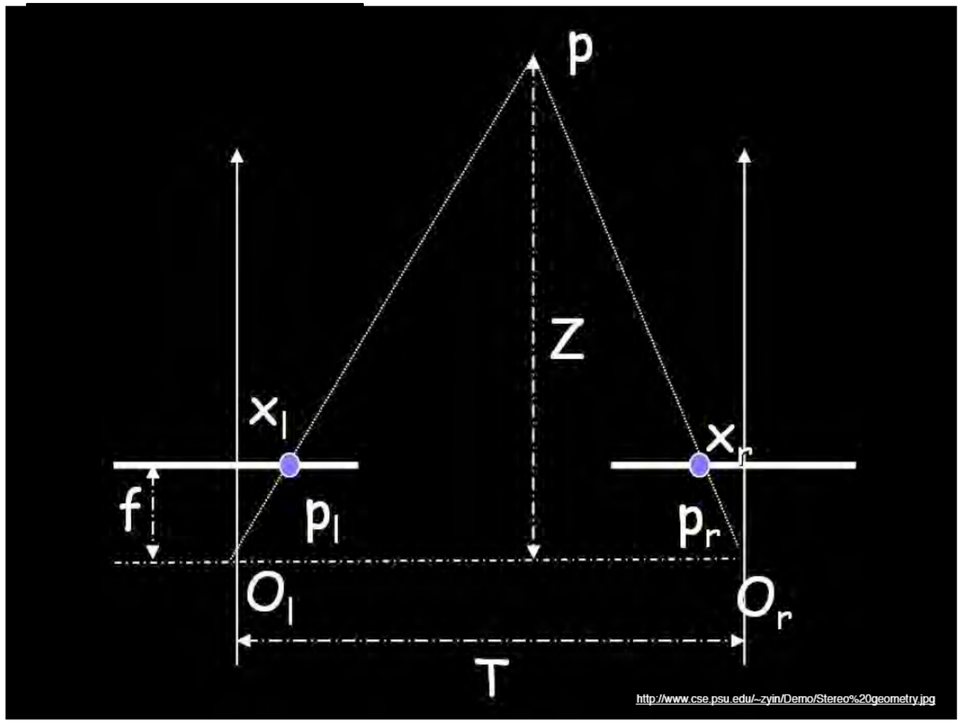

Similar triangles (p_l, P, p_r) and (O_l, P, O_r):

$$\frac{T + x_r - x_l}{Z - f} = \frac{T}{Z}$$

$$Z = f\frac{T}{x_l - x_r}$$

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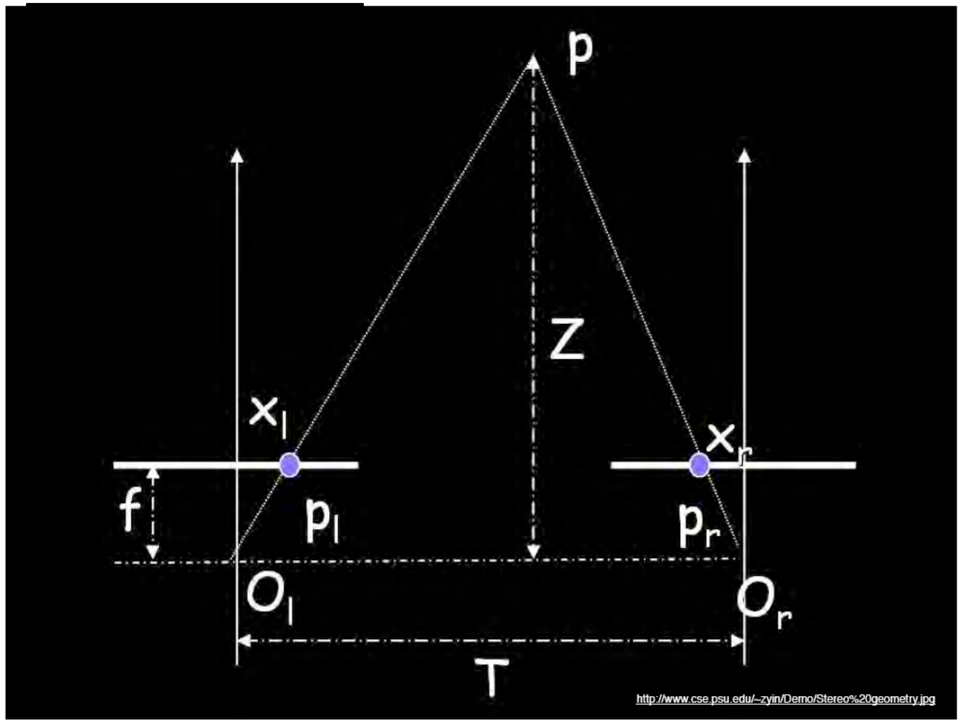

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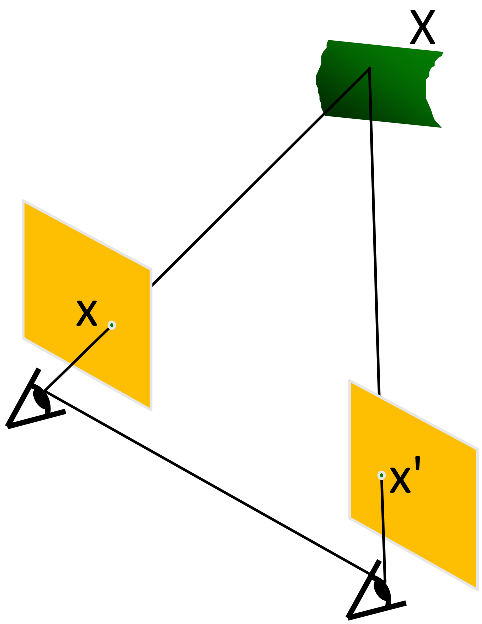
disparity (→ $x_l - x_r$)

Depth from disparity

- Goal: recover depth by finding image coordinate x’ that corresponds to x
- Sub-Problems
  1. Calibration: How do we recover the relation of the cameras (if not already known)?
  2. Correspondence: How do we search for the matching point x’?



Depth from disparity

image I(x,y)

Disparity map D(x,y)

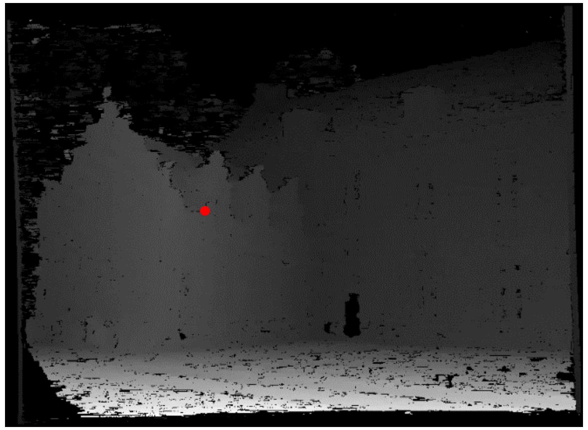
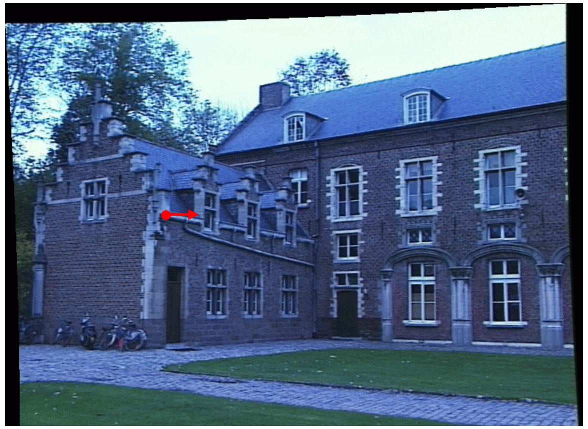


image I´(x´,y´)



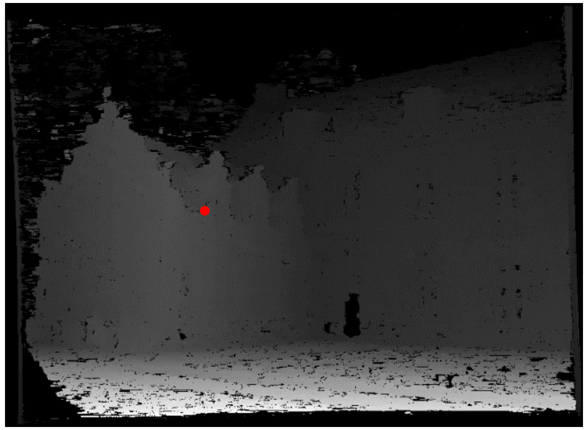
$$(x',y')=(x+D(x,y),\ y)$$

James Hays

Depth from disparity

image I(x,y)

Disparity map D(x,y)



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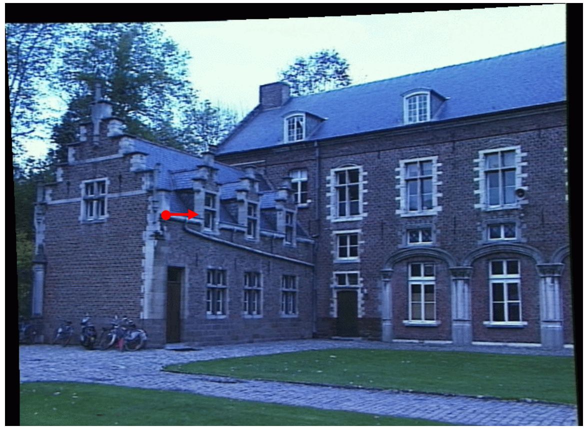

$$(x',y')=(x+D(x,y),\ y)$$

If we could find the **corresponding points** in two images, we could **estimate relative depth**…

James Hays

What do we need to know?

1. Calibration for the two cameras.
    1. Intrinsic matrices for both cameras (e.g., f)
    2. Baseline distance T in parallel camera case
    3. R, t in non-parallel case

2. Correspondence for every pixel.

Correspondence for every pixel.
Where do we need to search?

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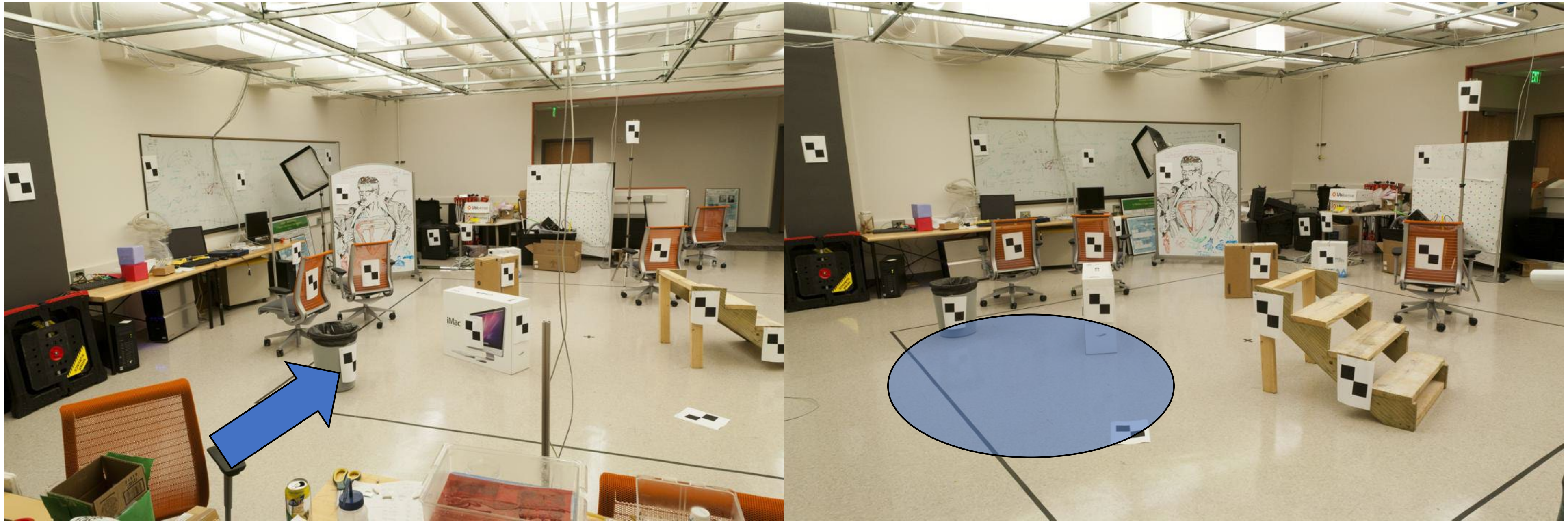

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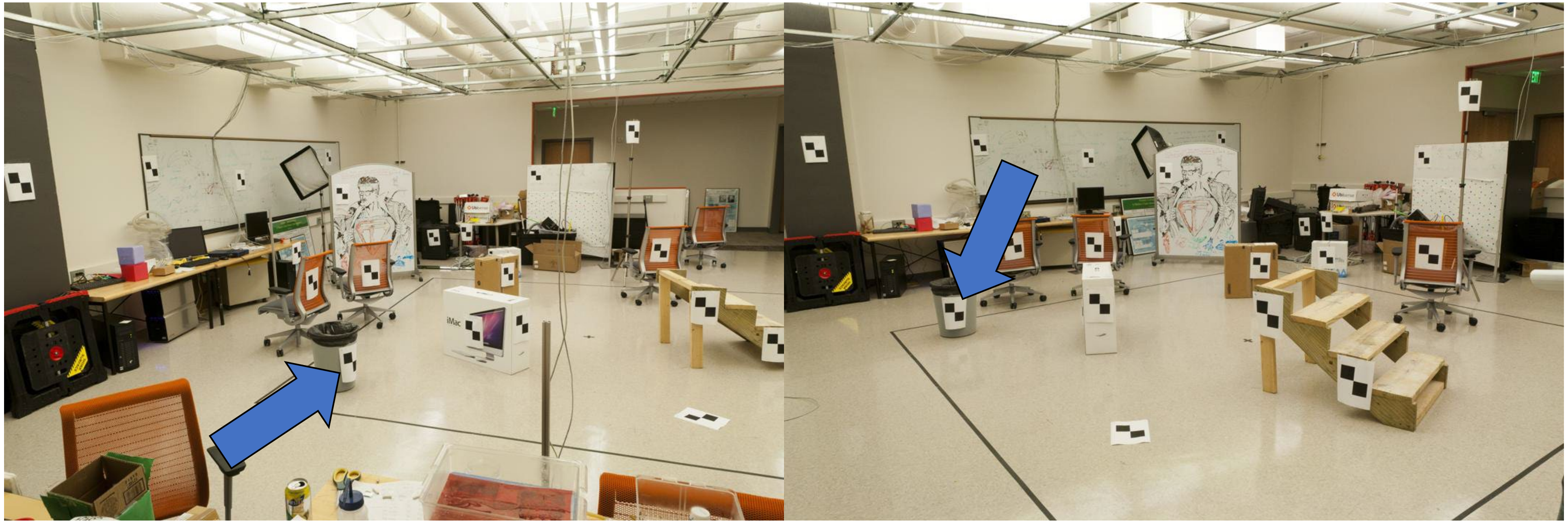

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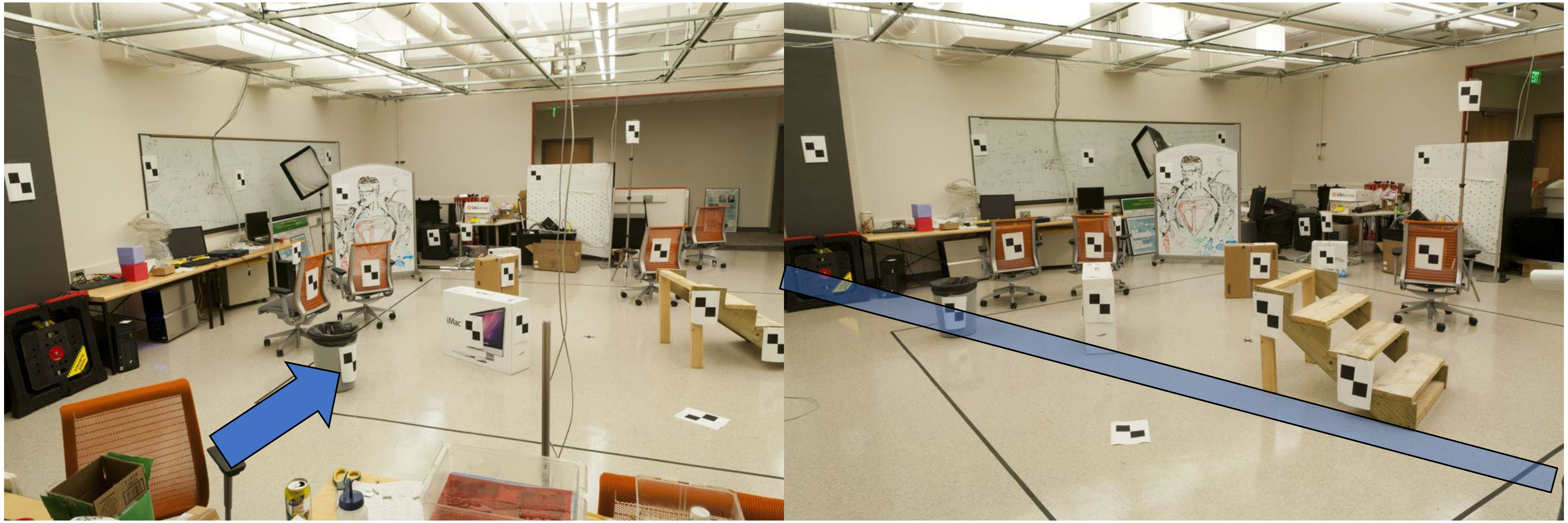
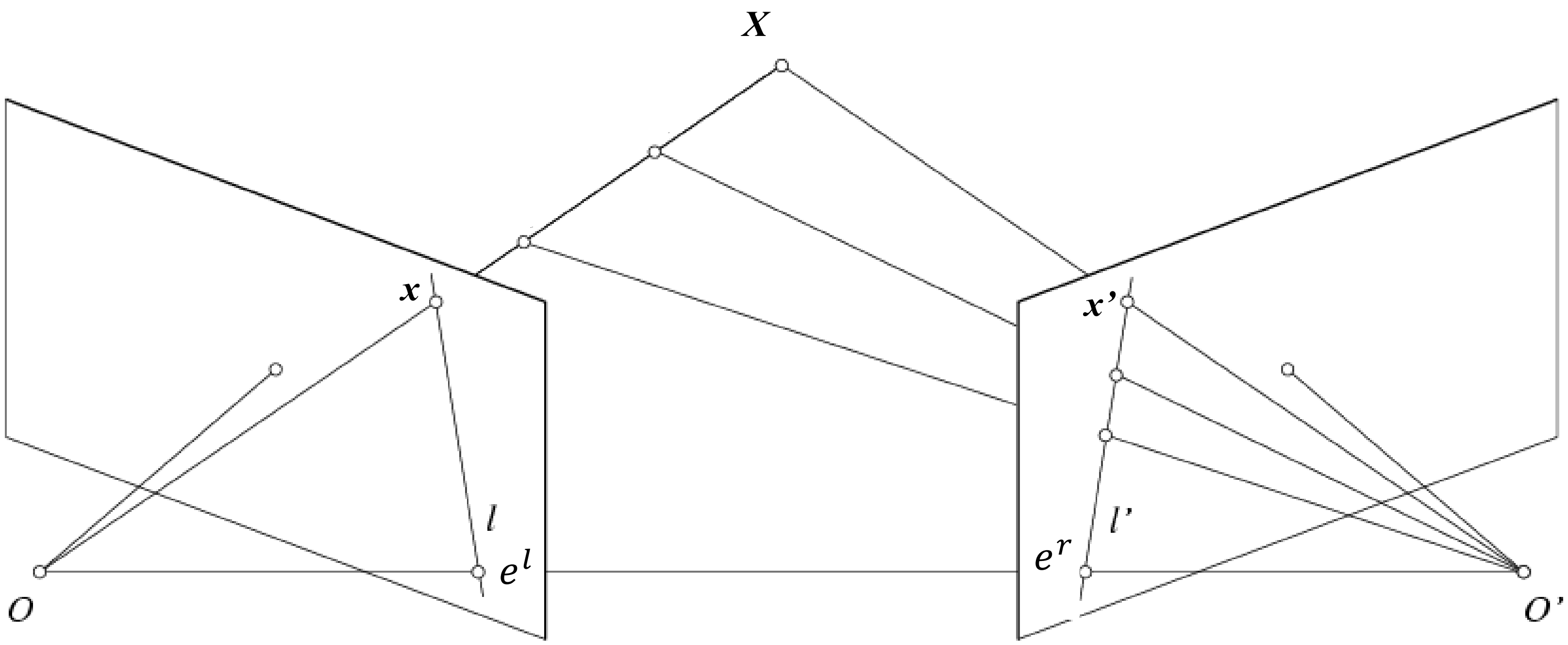


Wouldn’t it be nice to know where matches can live?

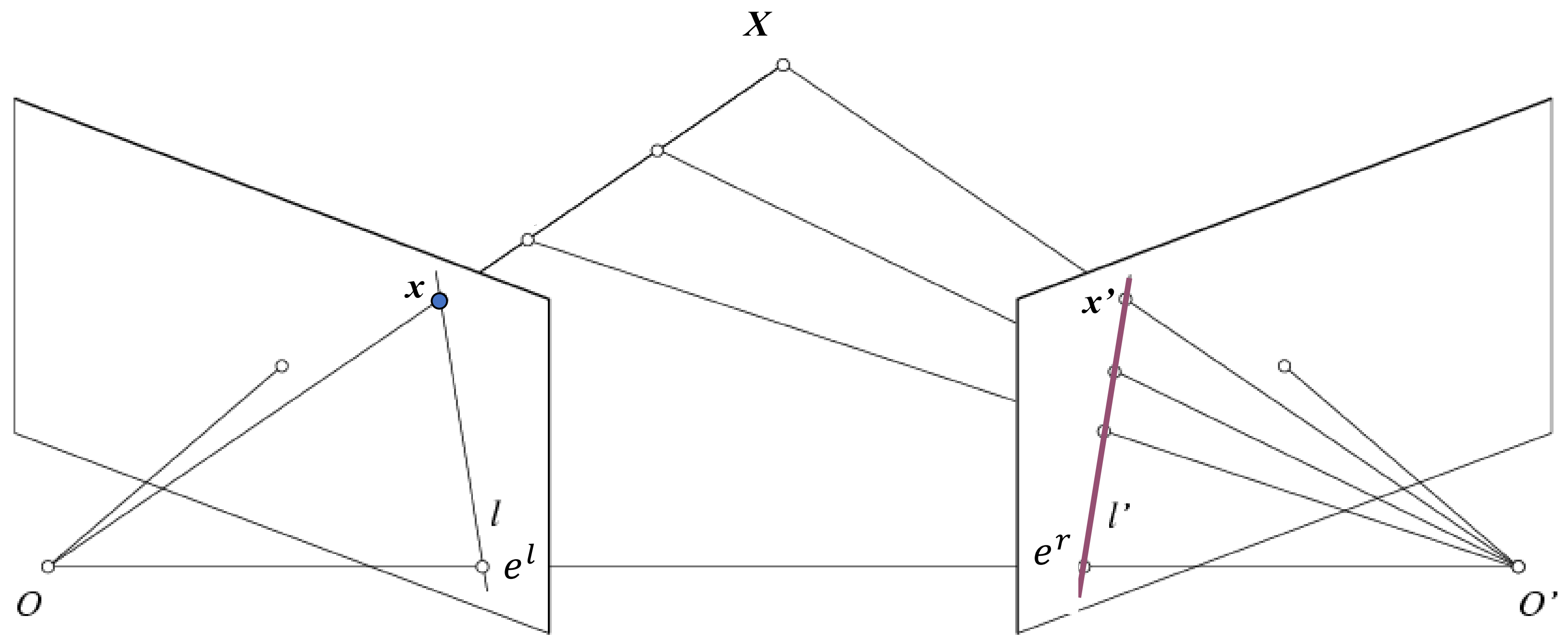
Epipolar geometry

Constrains 2D search to 1D

Key idea: Epipolar constraint

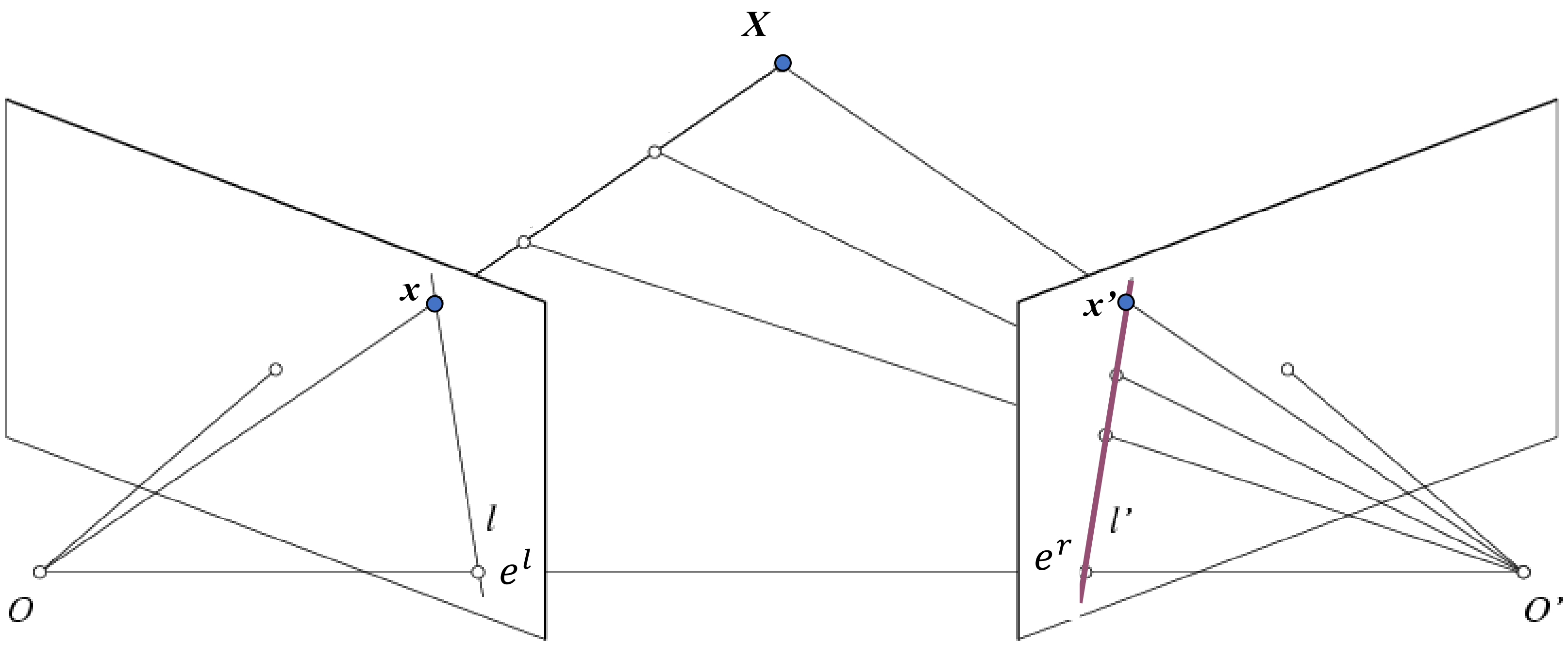


Key idea: Epipolar constraint



Potential matches for *x* have to lie on the corresponding line *l'*.

Key idea: Epipolar constraint



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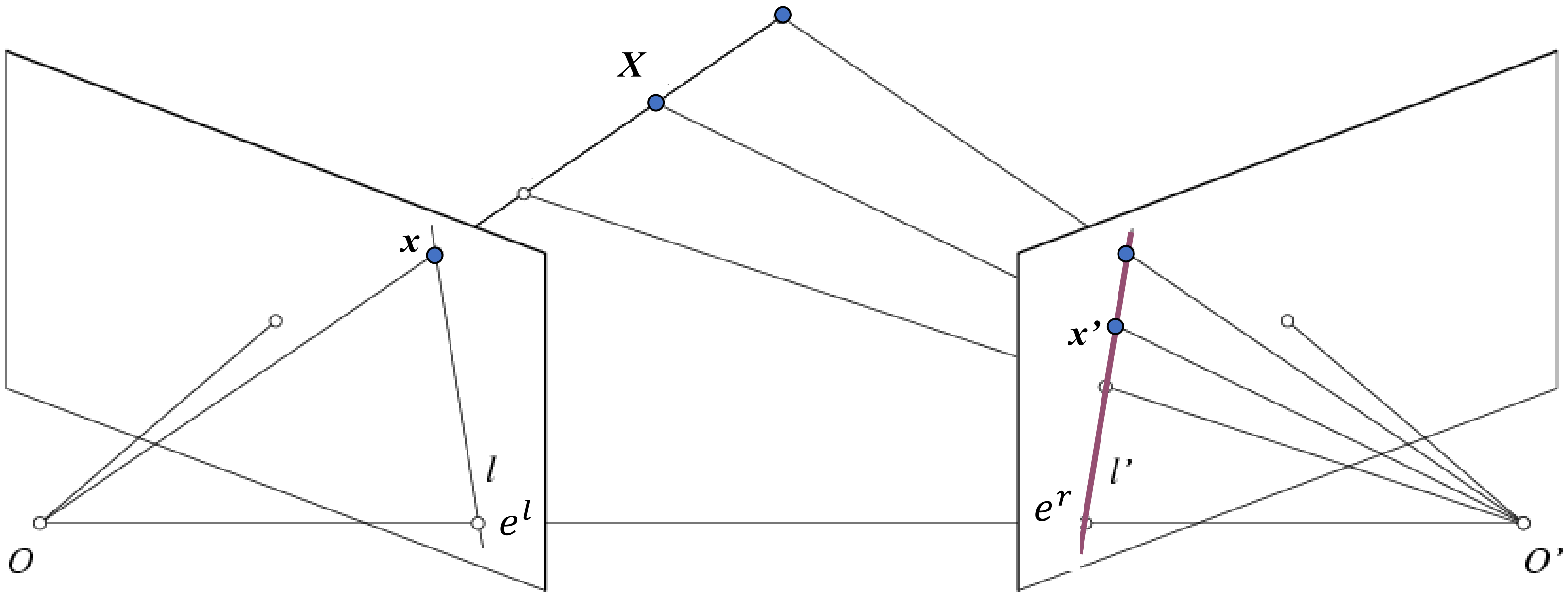

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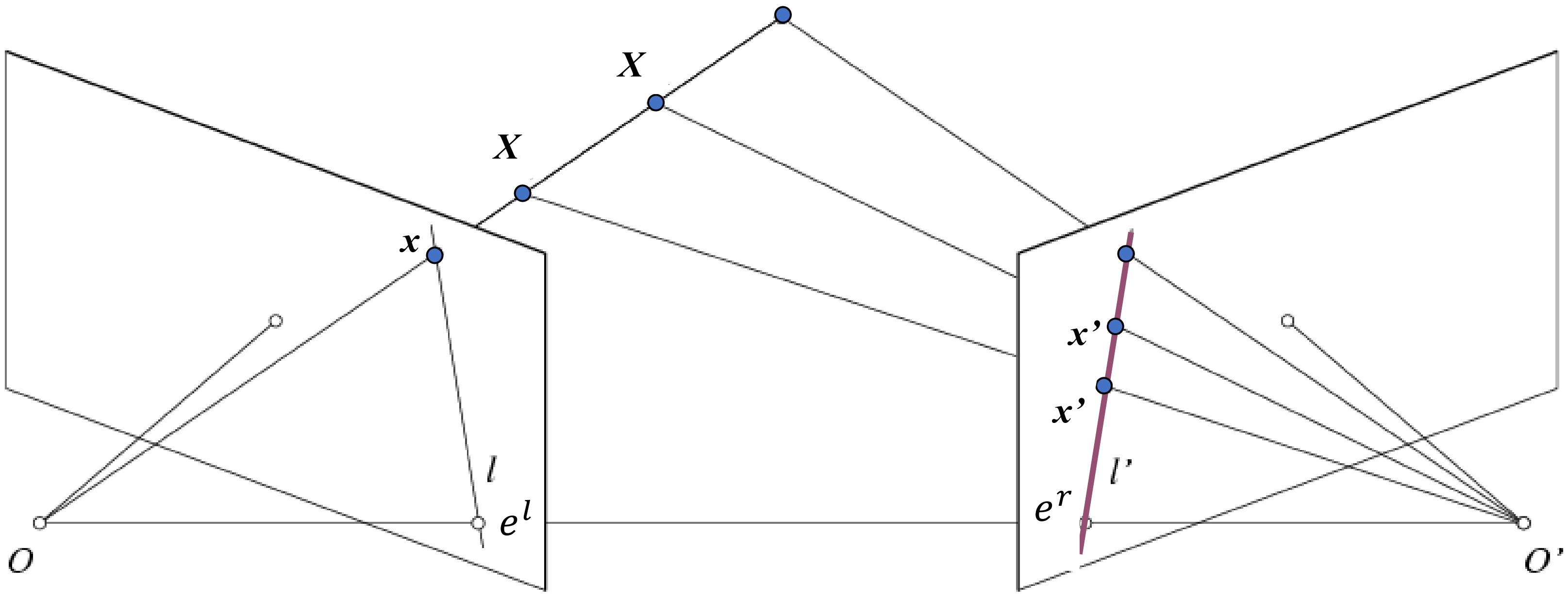

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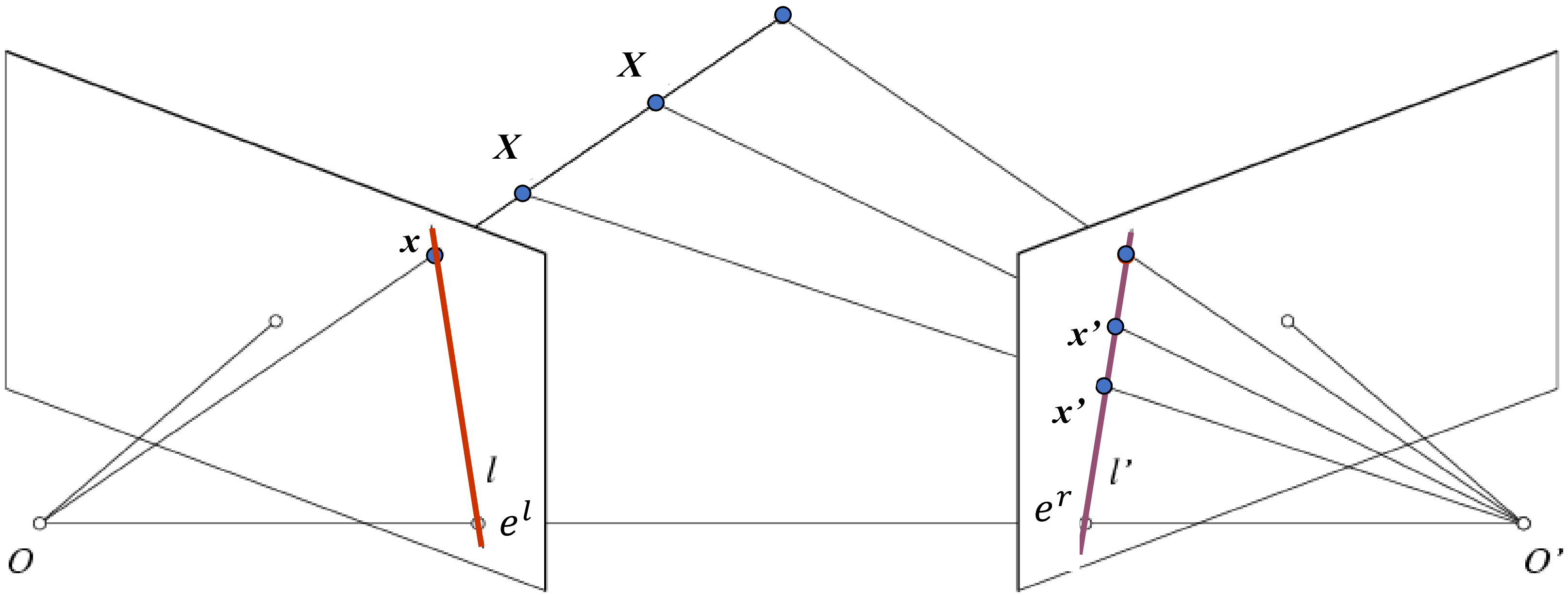

Potential matches for *x'* have to lie on the corresponding line *l*.

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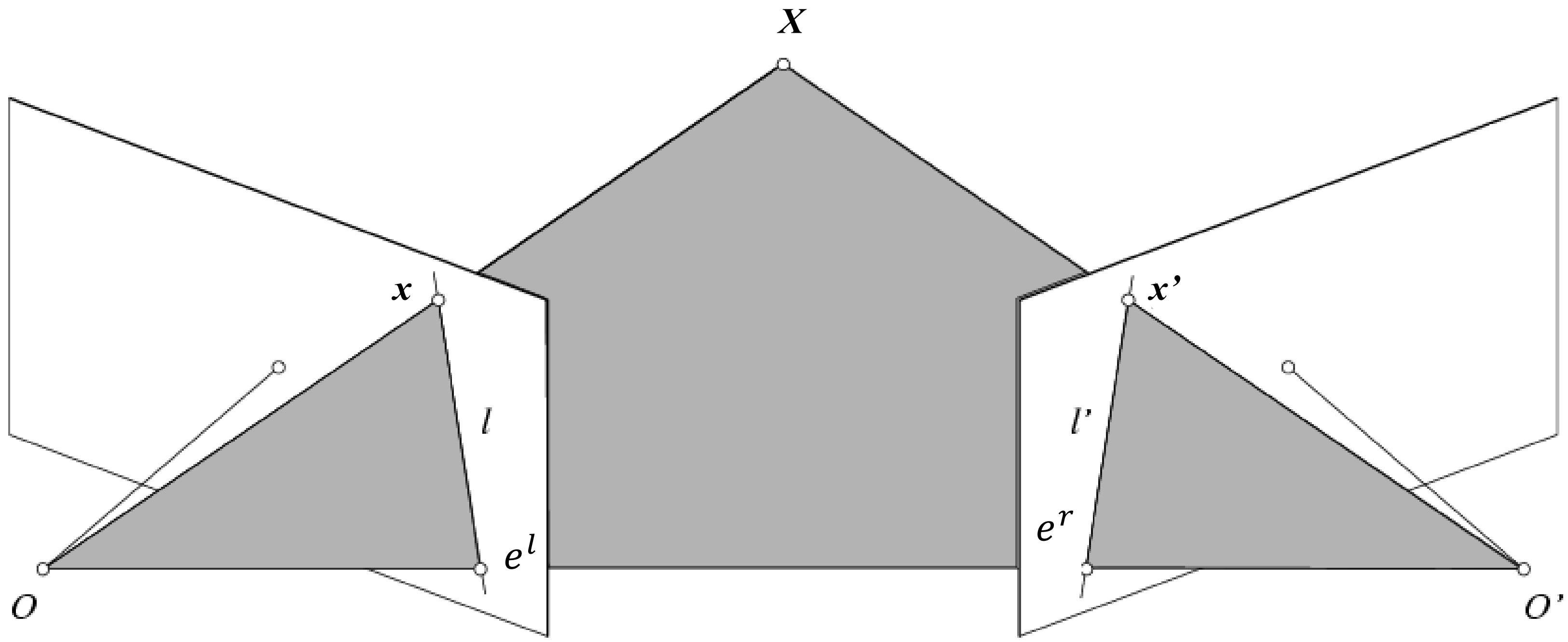
Epipolar geometry: notation



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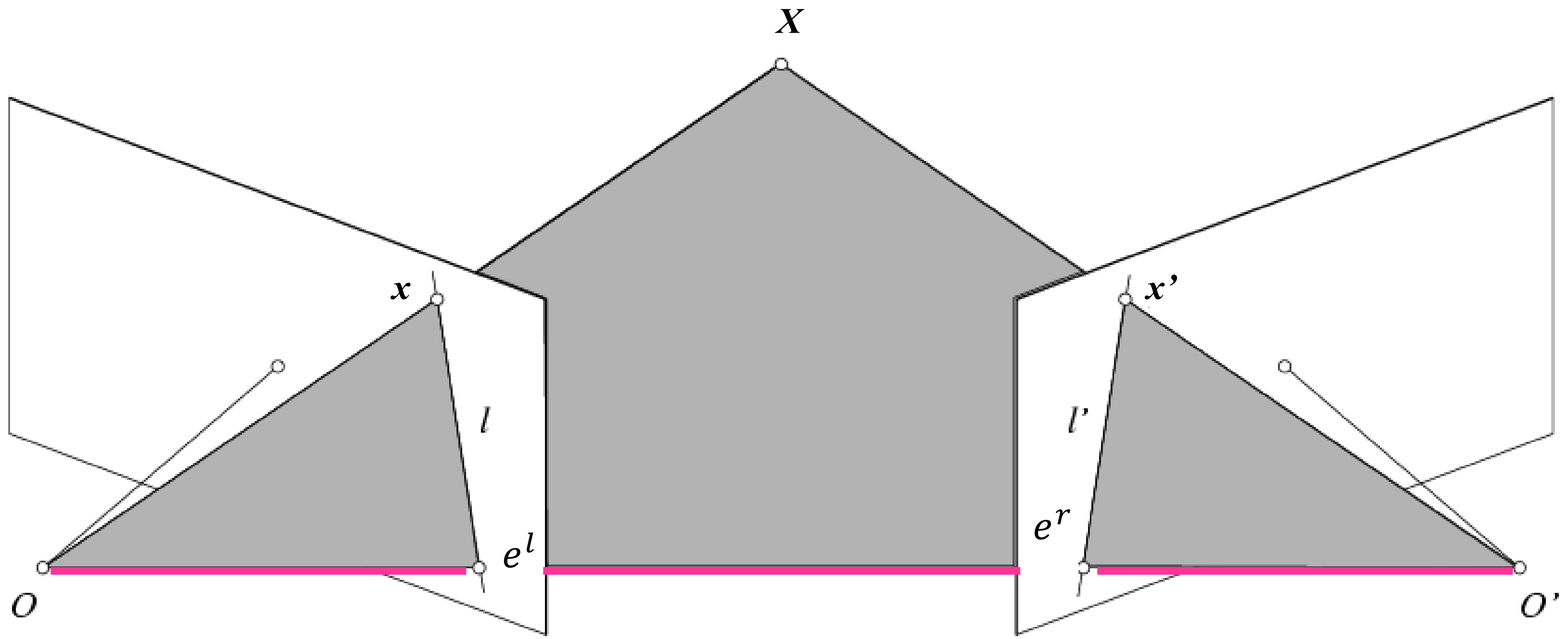
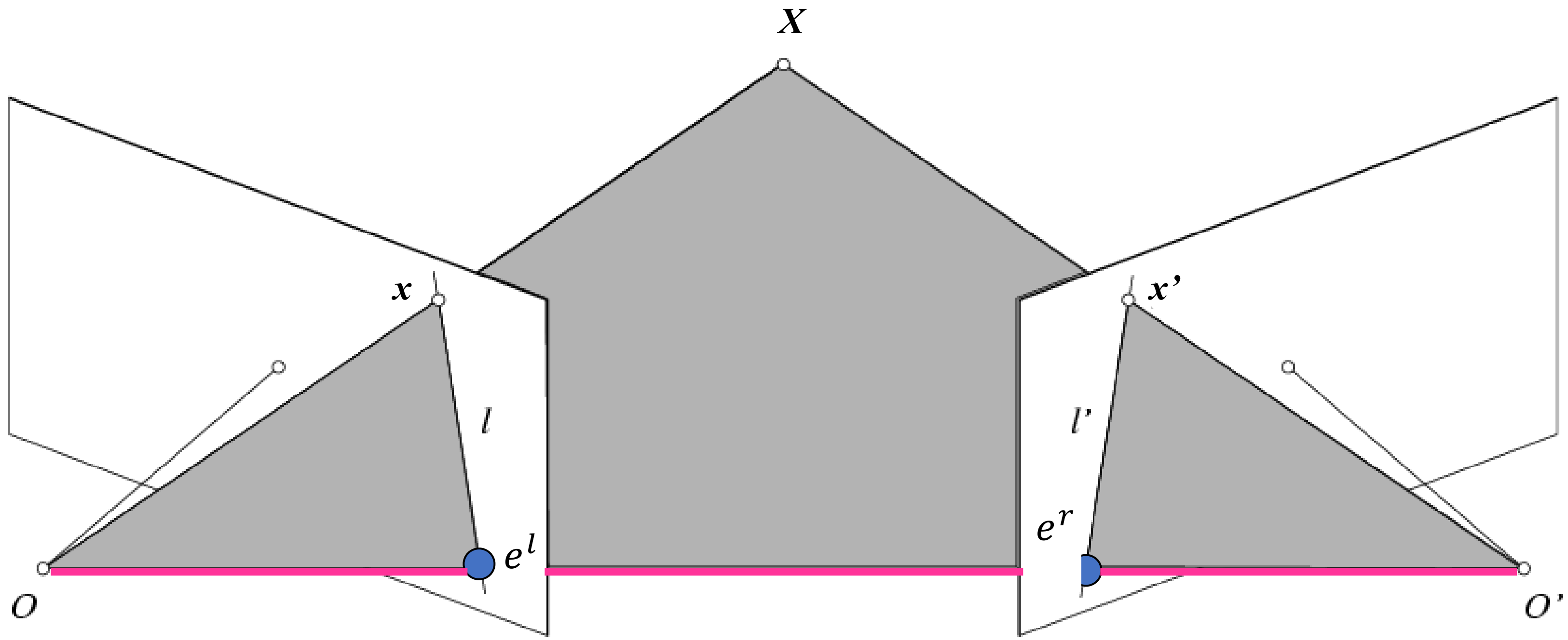


- **Baseline** – line connecting the two camera centers

Epipolar geometry: notation



- **Baseline** – line connecting the two camera centers
- **Epipoles**

= intersections of baseline with image planes
= projections of the other camera center

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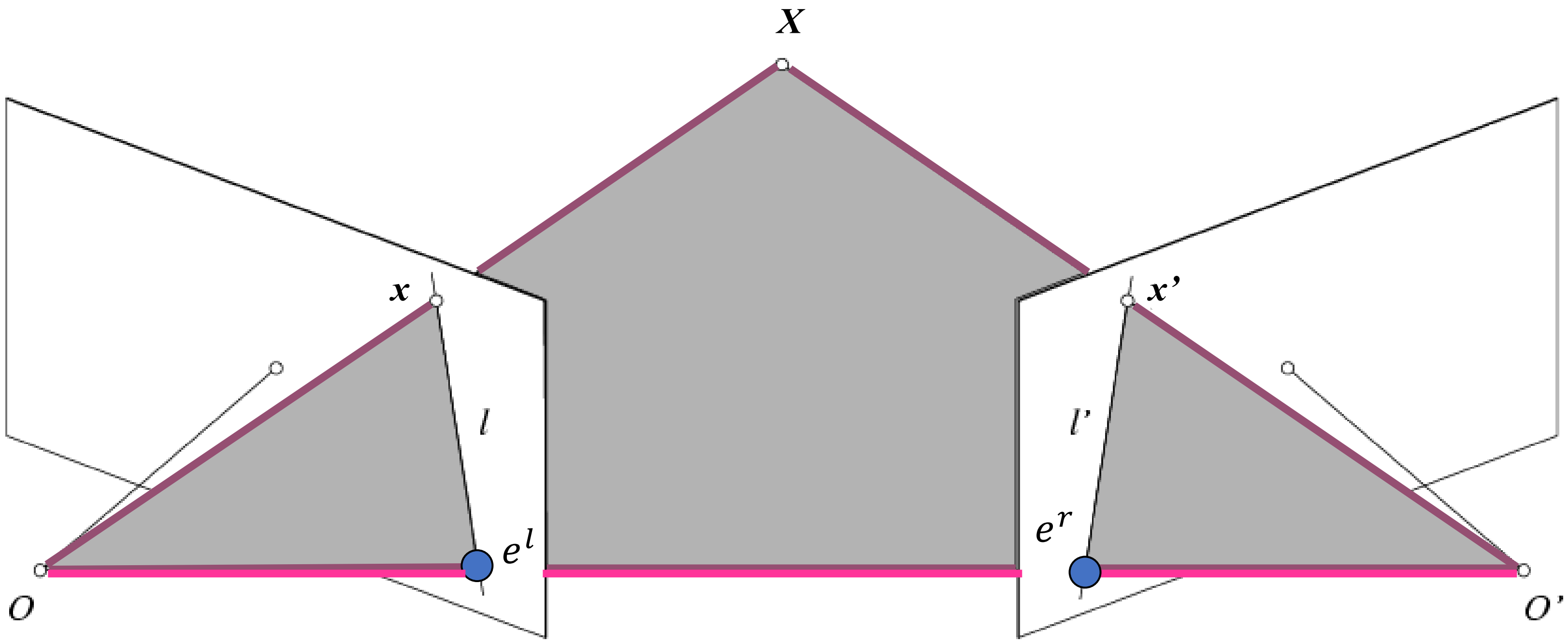
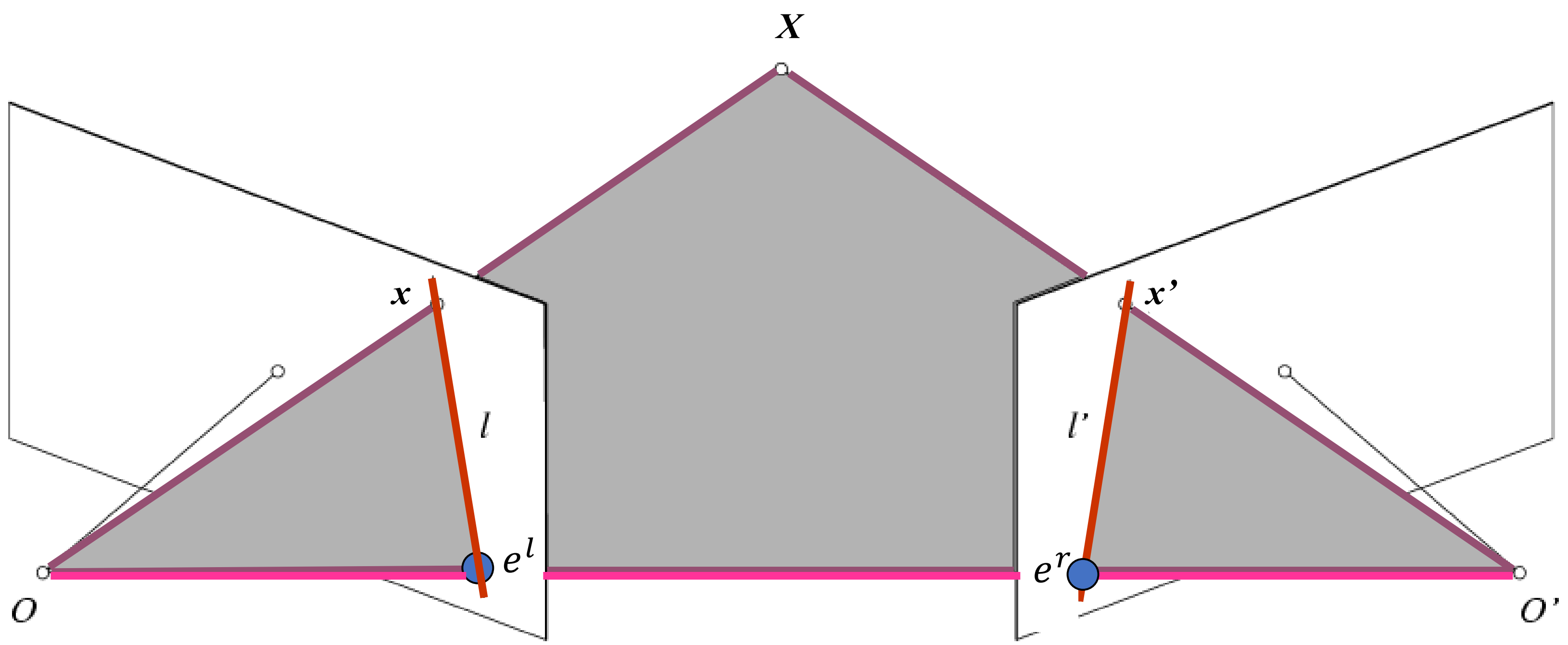


- **Baseline** – line connecting the two camera centers
- **Epipoles**
  = intersections of baseline with image planes
  = projections of the other camera center
- **Epipolar Plane** – plane containing baseline (1D family)

Epipolar geometry: notation



- **Baseline** – line connecting the two camera centers
- **Epipoles**

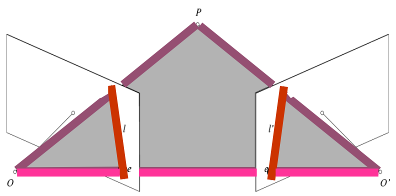
= intersections of baseline with image planes

= projections of the other camera center

- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)

Think Pair Share

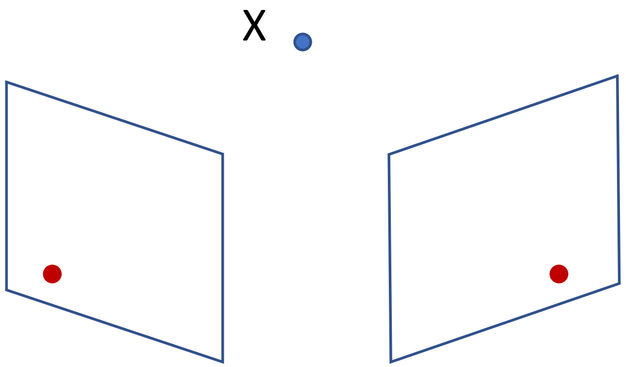
● = camera center



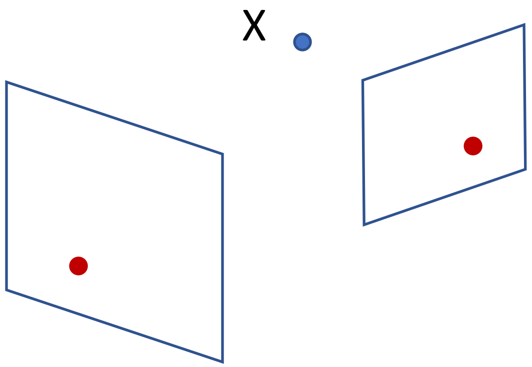
Where are the epipoles?

What do the epipolar lines look like?

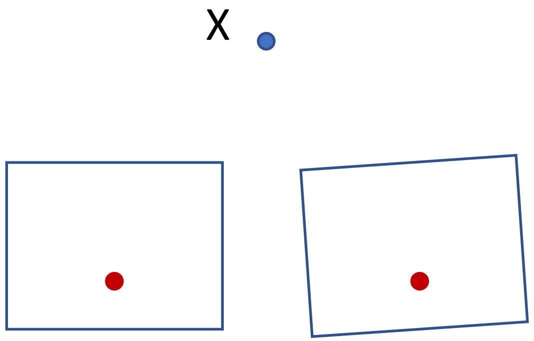
a)



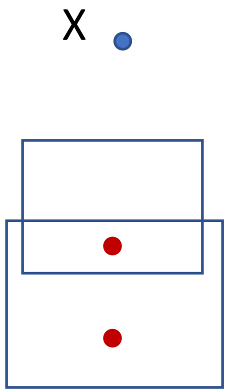
b)



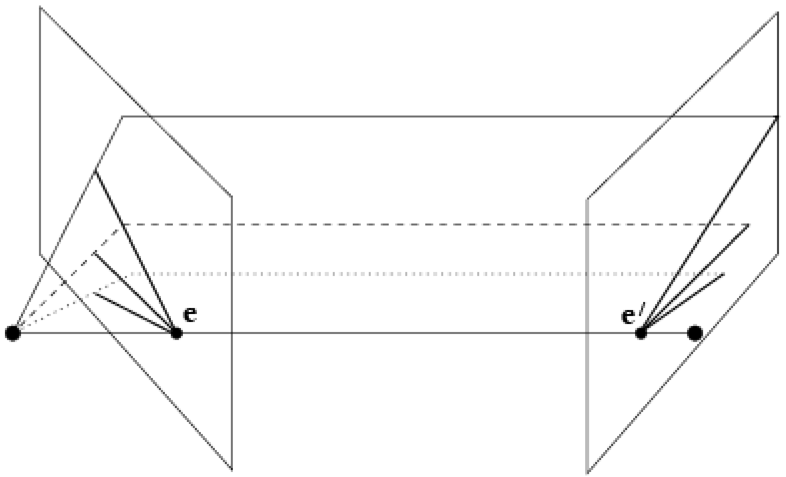
c)

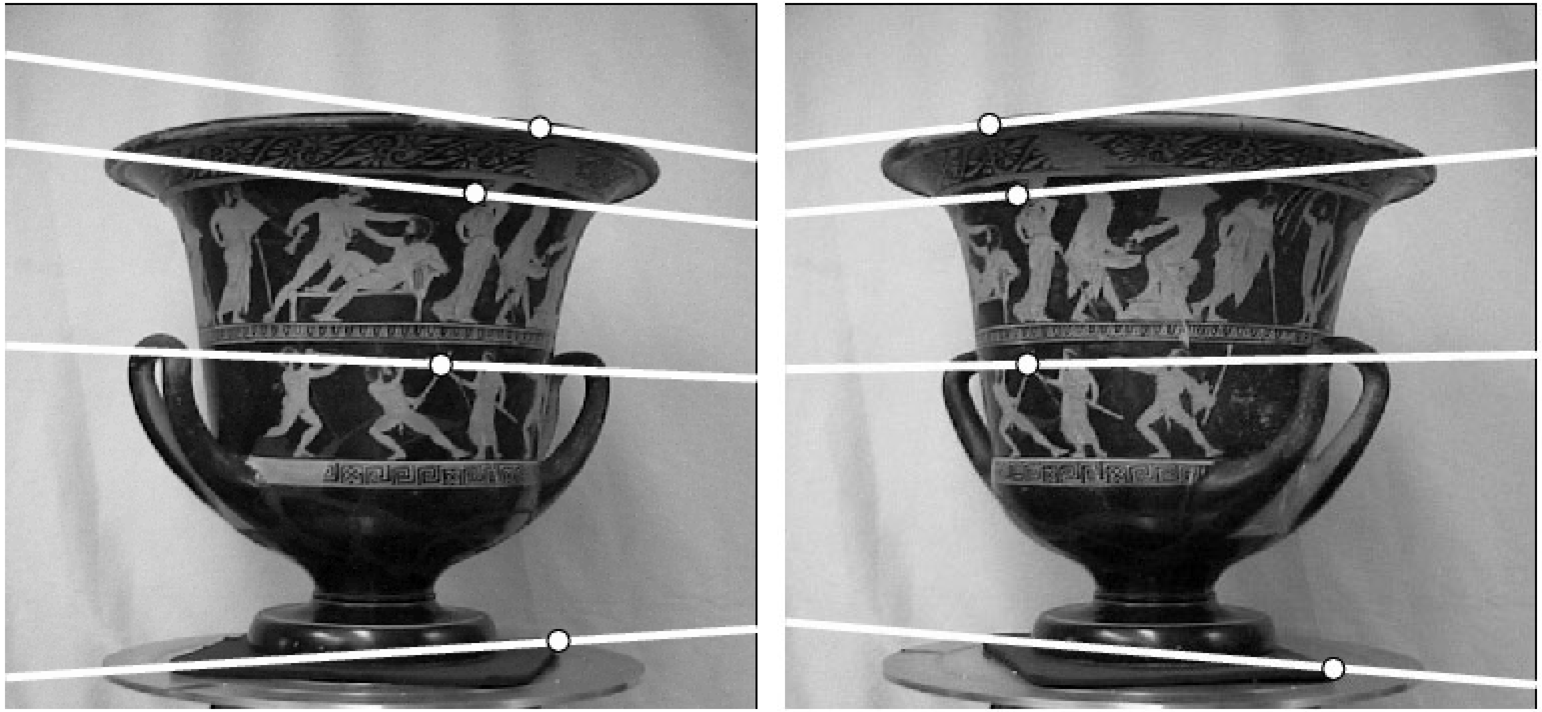


d)

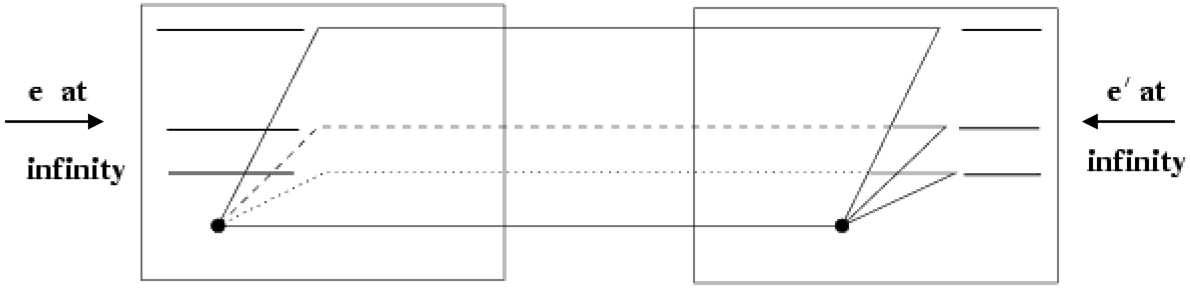


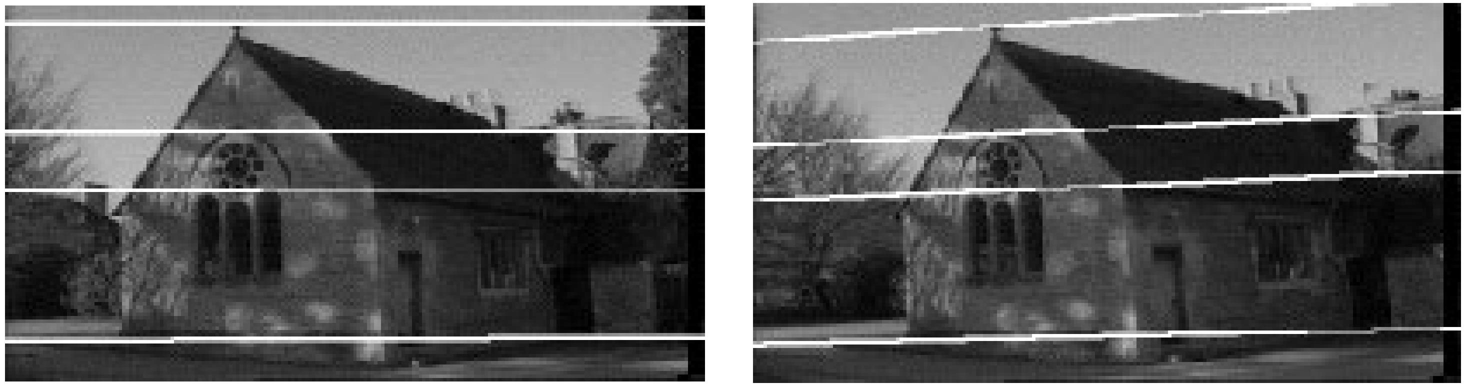
Example: Converging cameras



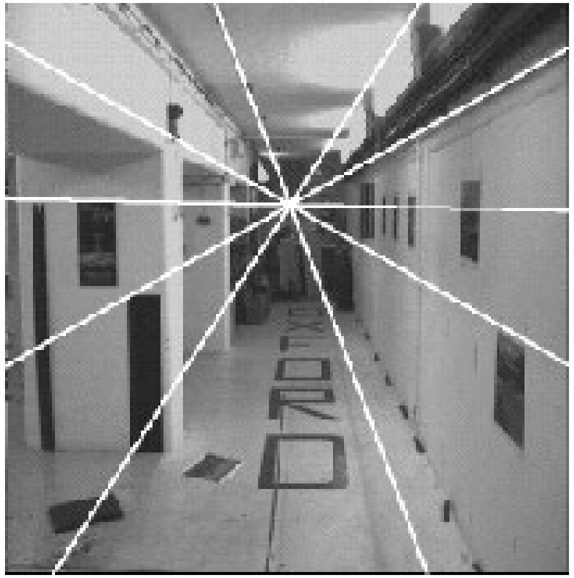


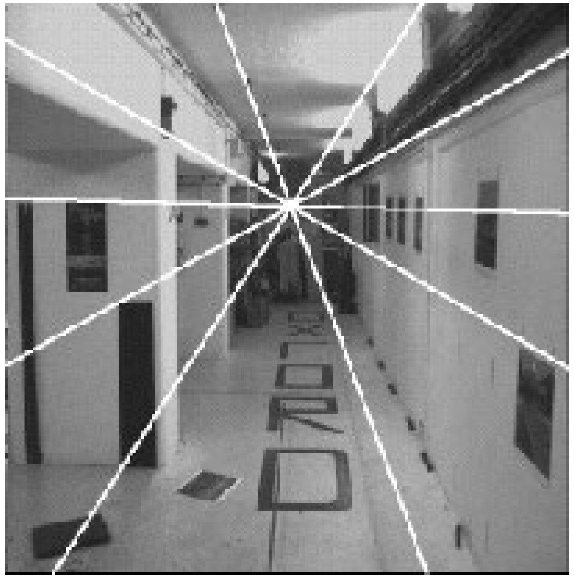
Example: Motion parallel to image plane

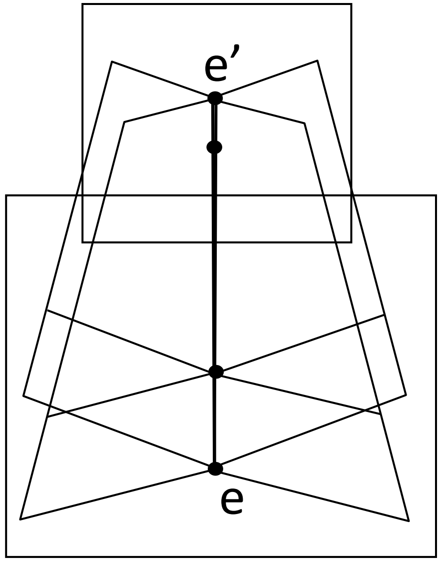




Example: Forward motion





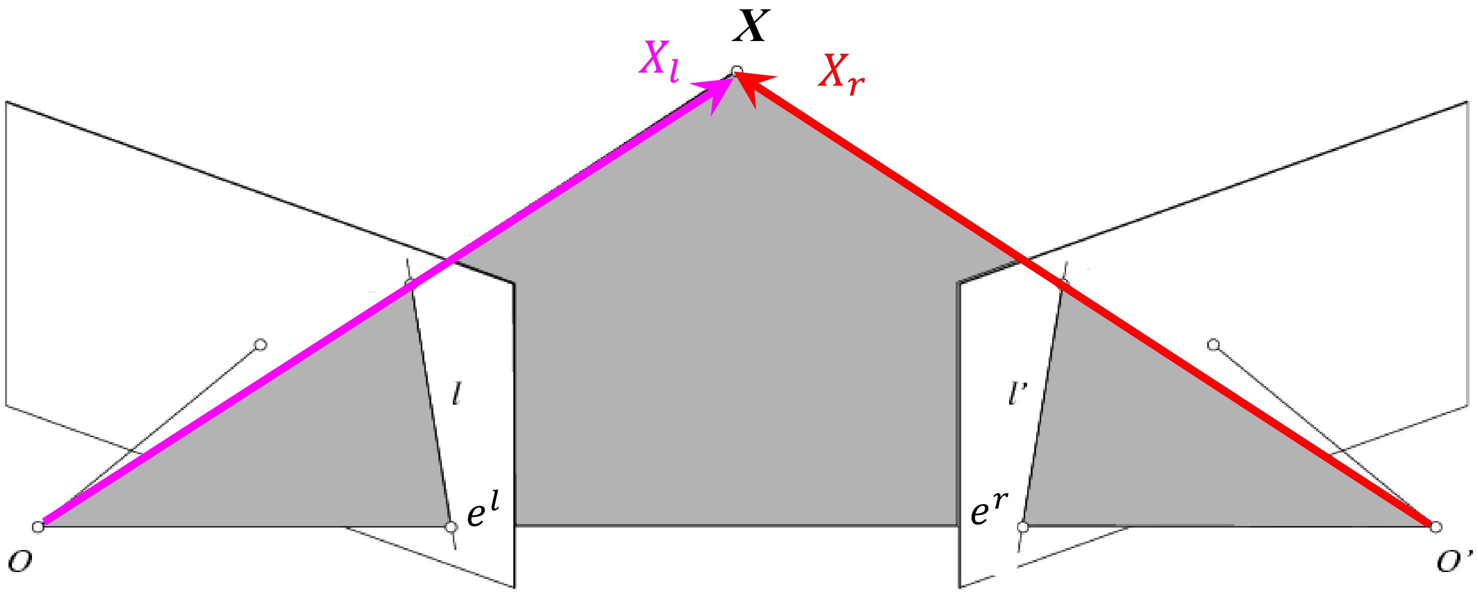


Epipole has same coordinates in both images.
Points move along lines radiating from e:
“Focus of expansion”

How to find epipolar line of a point?

- A little bit more math, but cool math
- Essential matrix and fundamental matrix

Essential matrix

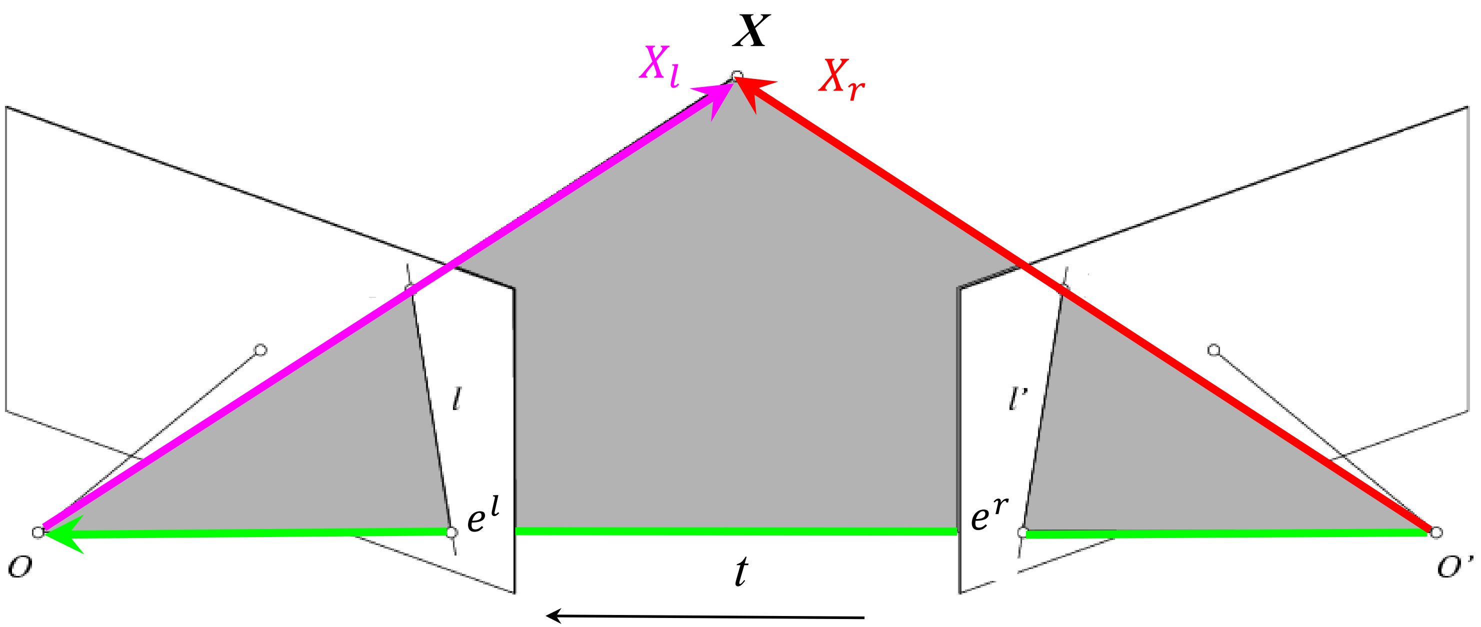


X_l: X in terms of Cartesian coordinate of left camera

X_r: X in terms of Cartesian coordinate of right camera

There exists E such that ${X_l}^T E X_r = 0$

Essential matrix



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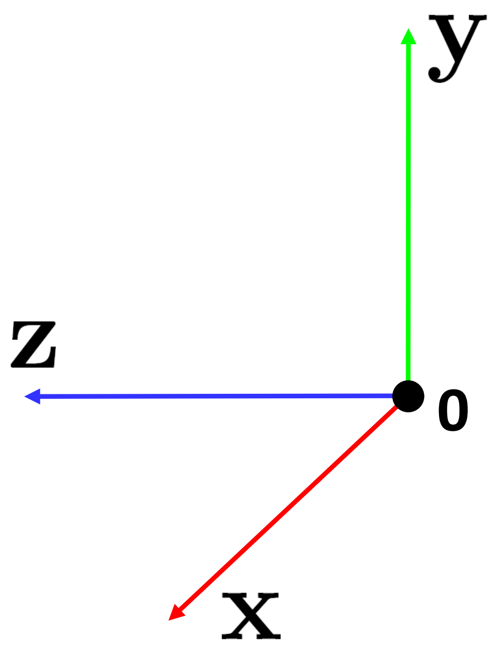
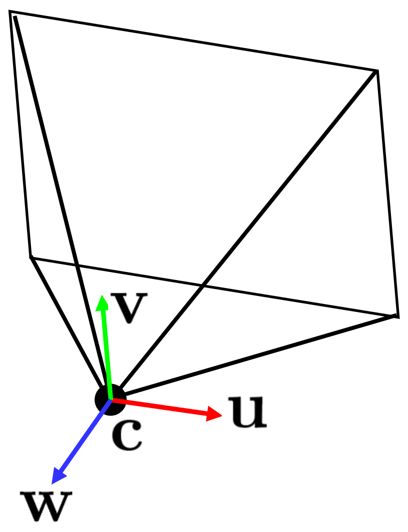
Review: change of coordinate

- How do we change coordinate from one camera to another?
  - (Center of projection at the origin, x-axis points right, y-axis points up, z-axis points backwards)





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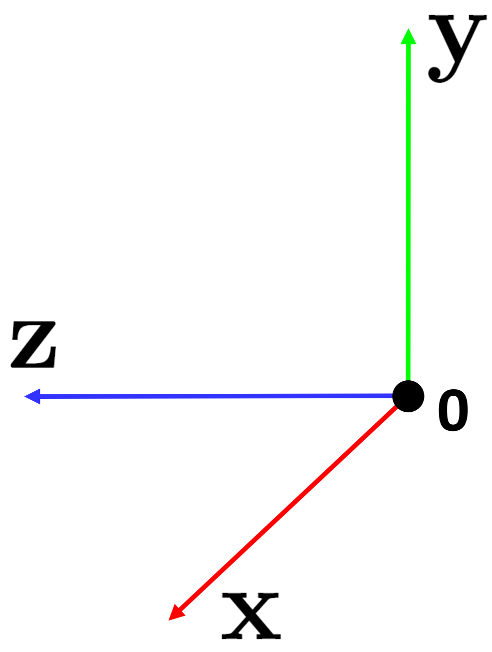
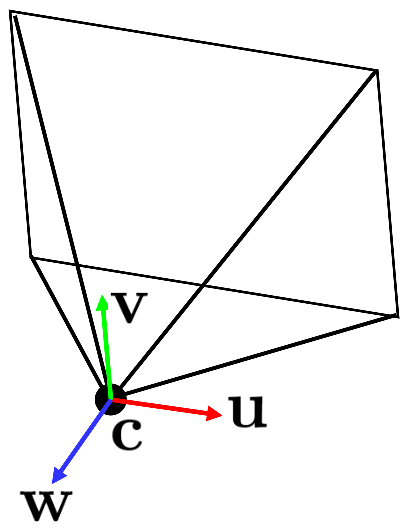


Step 1: Translate by -**c**



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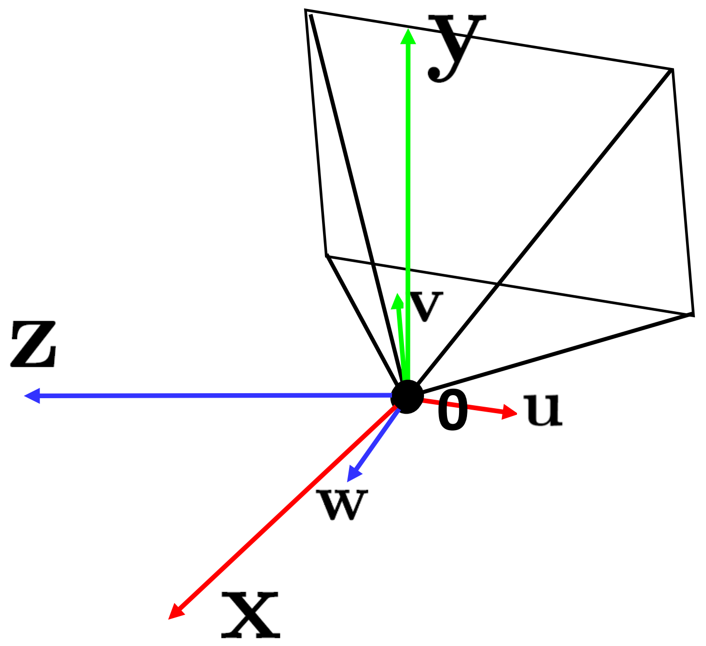

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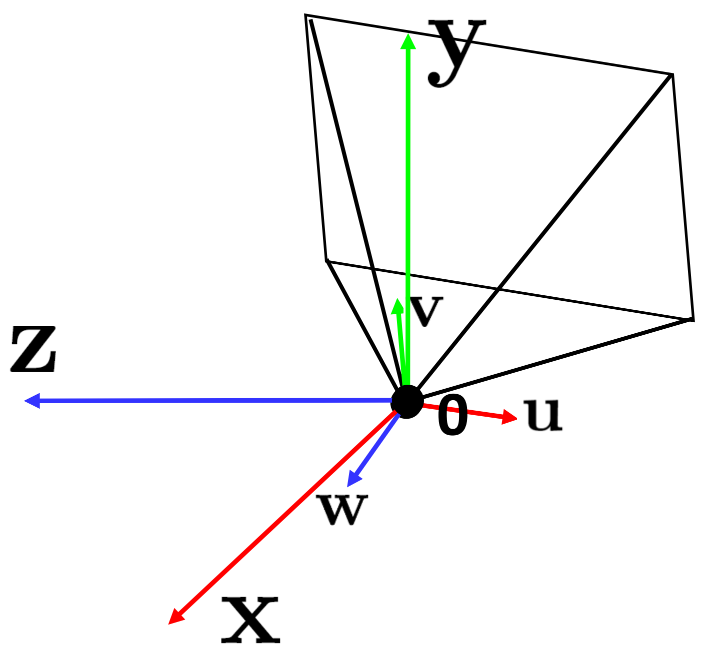

Step 1: Translate by -**c**

How do we represent translation as a matrix multiplication?

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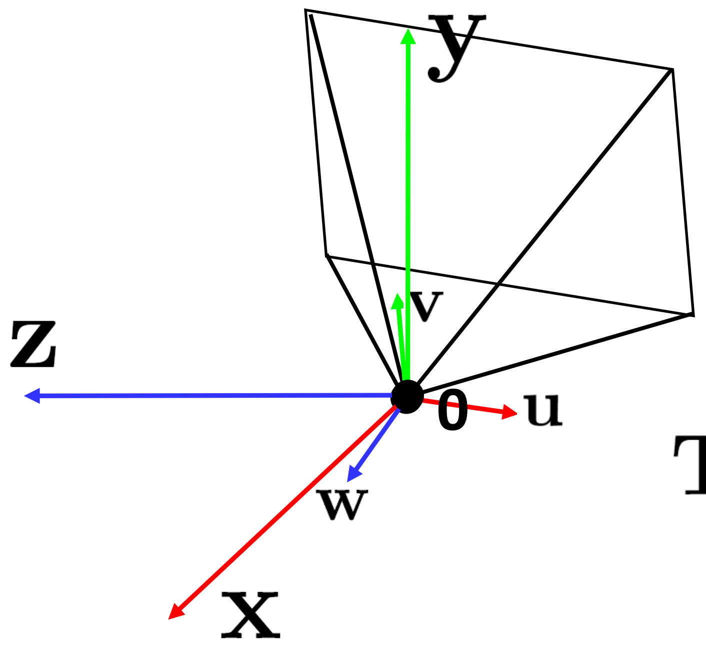

Step 1: Translate by -**c**

How do we represent translation as a matrix multiplication?

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{3\times 3} & -\mathbf{c} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

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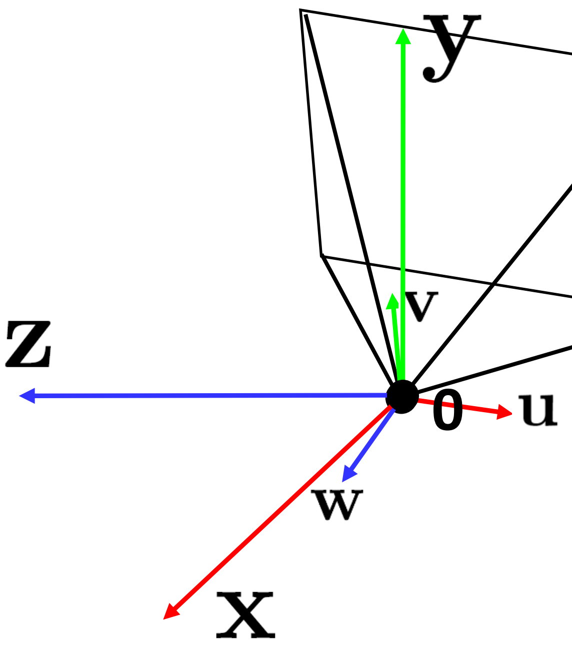

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}^T & \\ \mathbf{v}^T & \\ \mathbf{w}^T & \\ & 1 \end{bmatrix}$$

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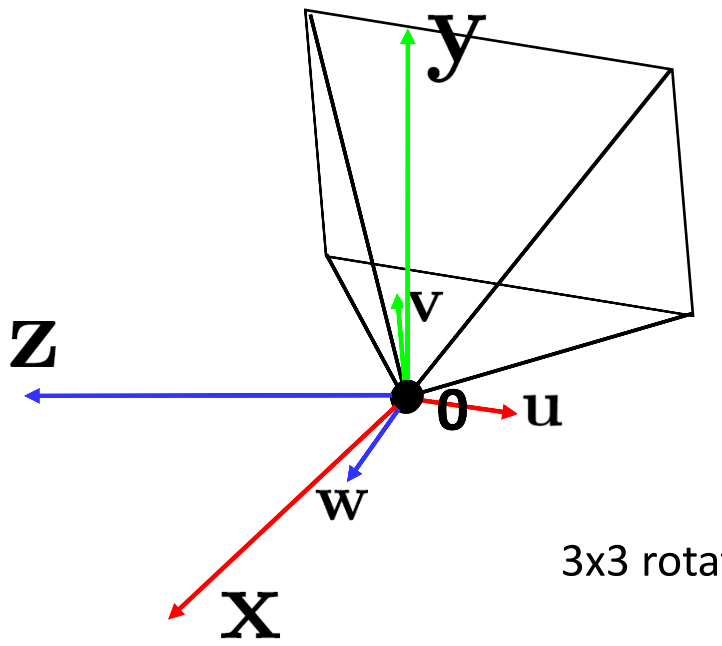

Step 1: Translate by **-c**

Step 2: Rotate by **R**

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}^T & \\ \mathbf{v}^T & \\ \mathbf{w}^T & \\ & 1 \end{bmatrix}$$

3x3 rotation matrix

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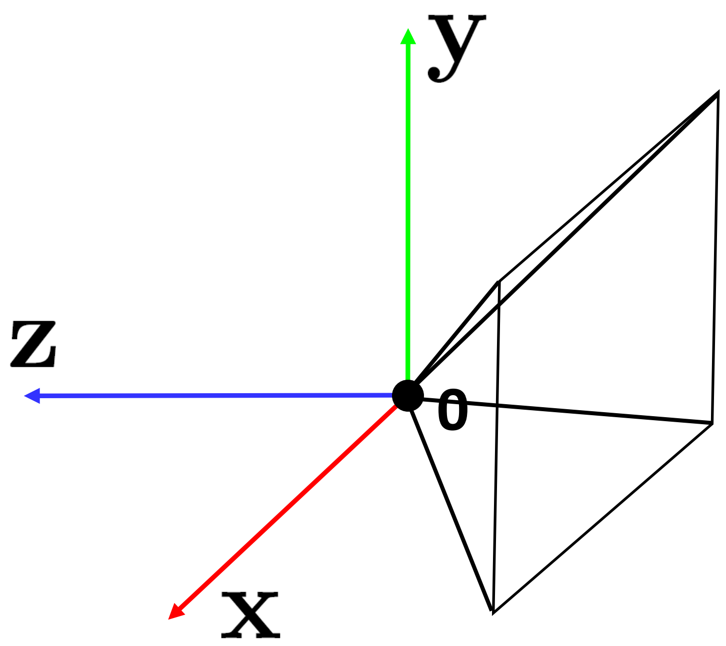
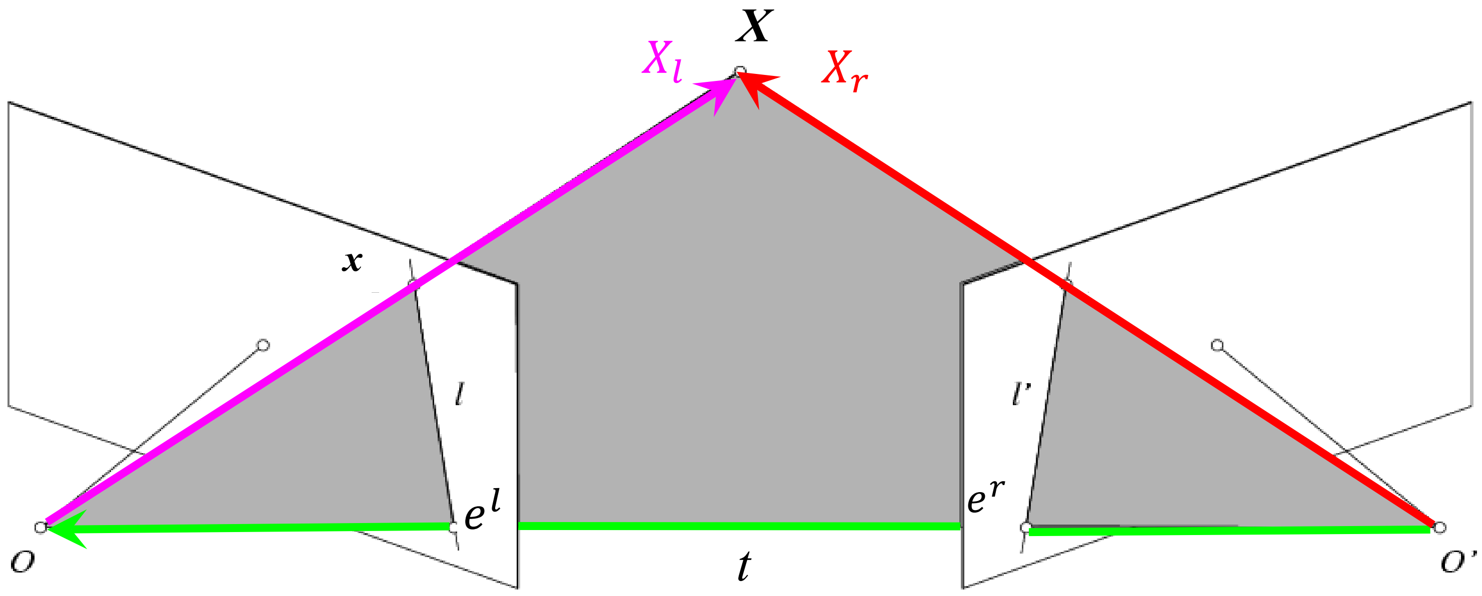


Step 1: Translate by -**c**

Step 2: Rotate by **R**

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}^T & \\ \mathbf{v}^T & \\ \mathbf{w}^T & \\ & 1 \end{bmatrix}$$

Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from left to right camera, then $X_l = R(X_r - t)$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right]^\top$$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right]^{\mathsf{T}}$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right]^\top$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[\mathbf{u}]_\times} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right]^\top$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[\mathbf{u}]_\times} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= [\mathbf{u}]_\times \mathbf{v}$$

Cross product in matrix representation

- Let $\mathbf{u} = [u_1, u_2, u_3]^T, \mathbf{v} = [v_1, v_2, v_3]^T$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

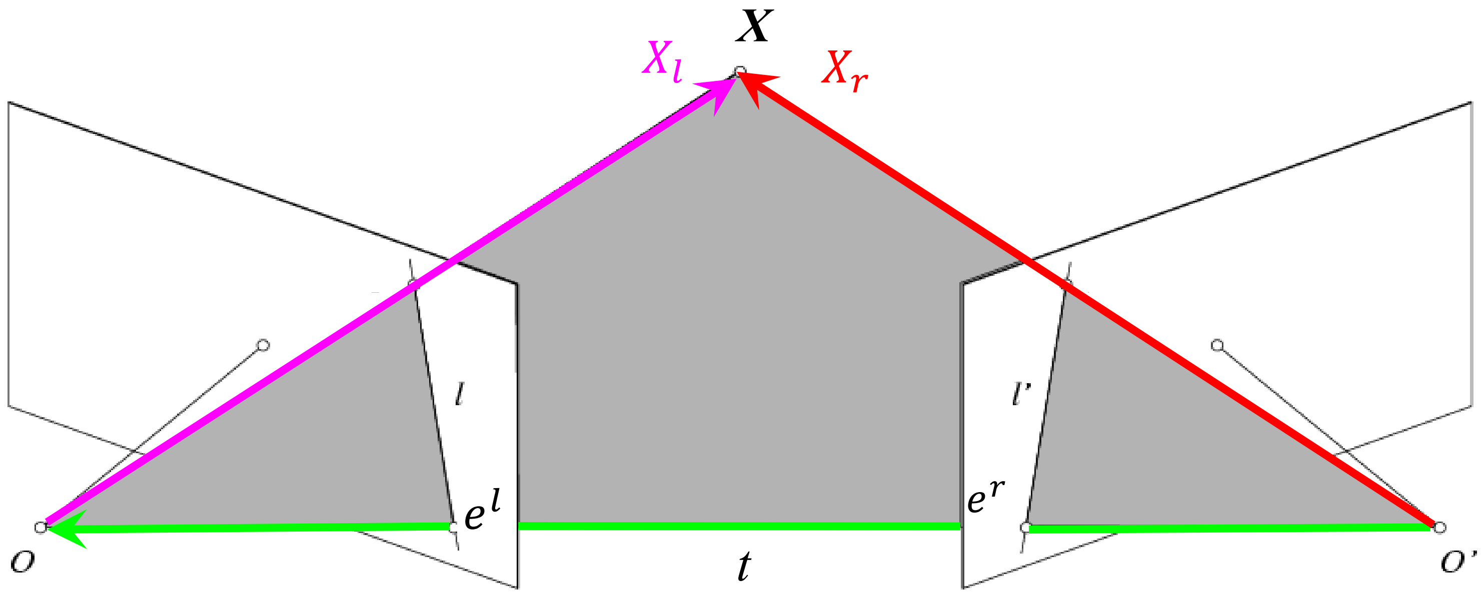
$$= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right]^{\mathsf{T}}$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[\mathbf{u}]_\times} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= [\mathbf{u}]_\times \mathbf{v}$$

N.B. $\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Therefore, $\operatorname{rank}([\mathbf{u}]_\times) \leq 2$

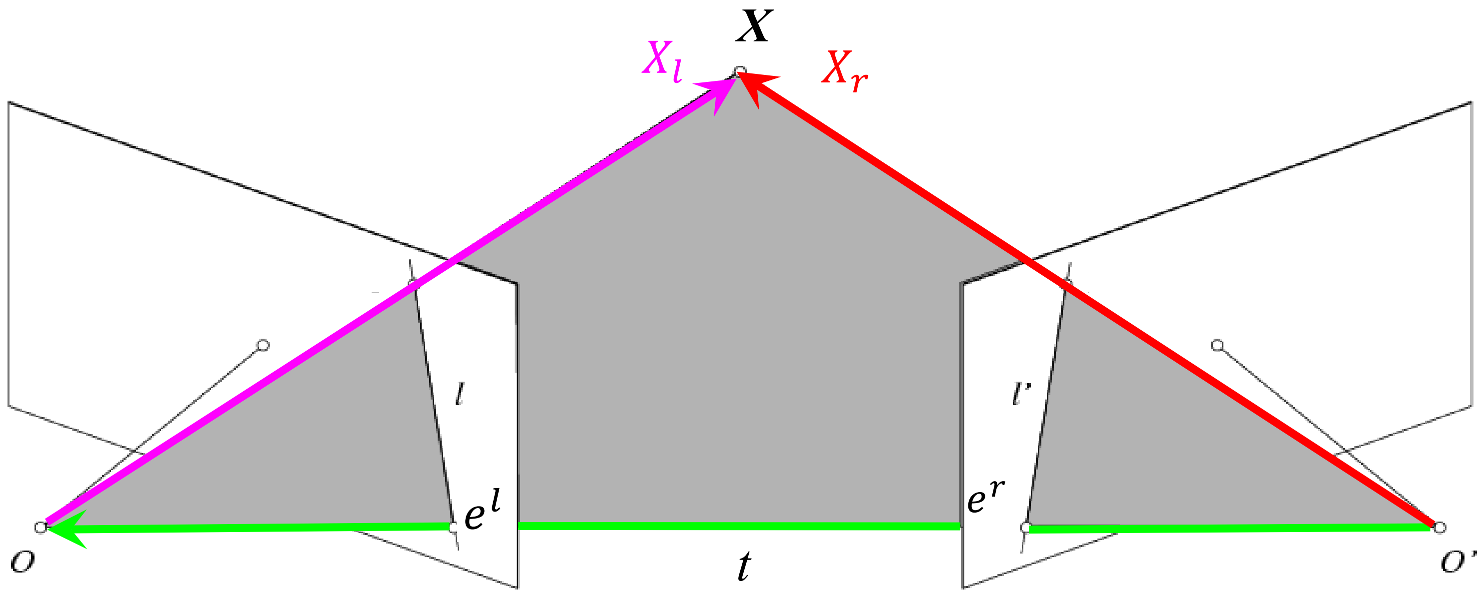
Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

Let E be $R[t]_{\times}$, $X_l^T E X_r$

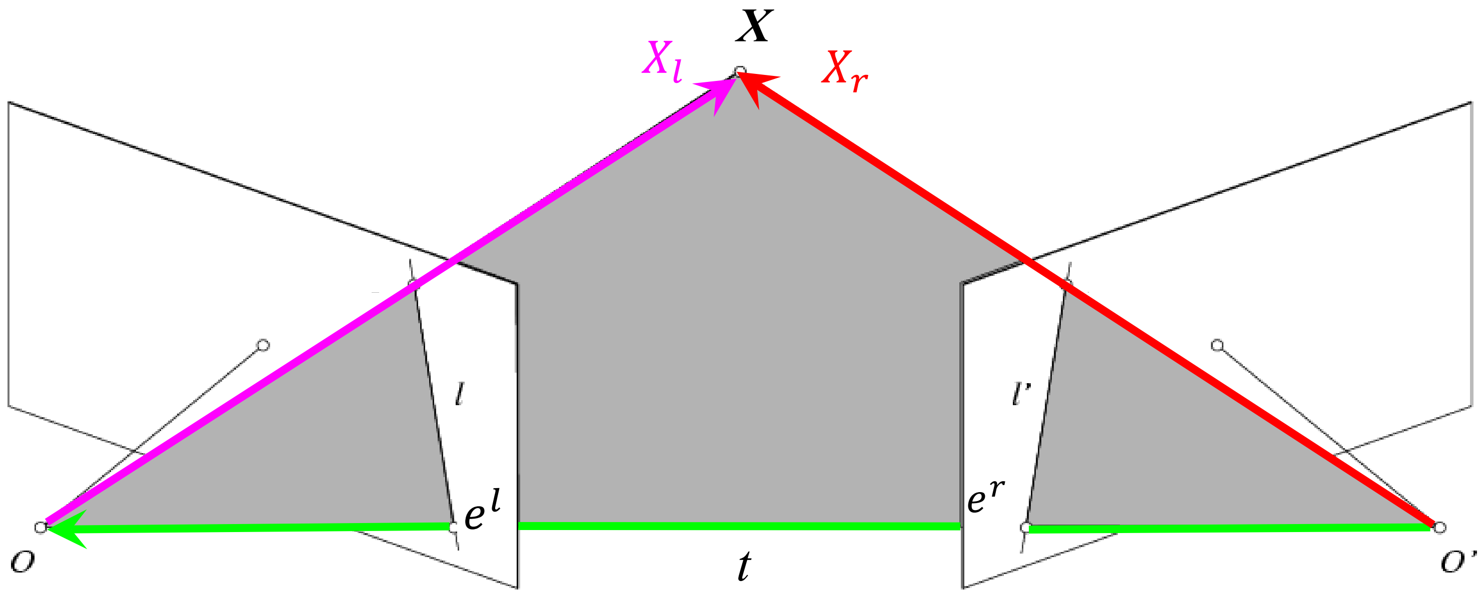
Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

Let E be $R[t]_\times$, $X_l^T E X_r = X_l^T (R[t]_\times X_r)$

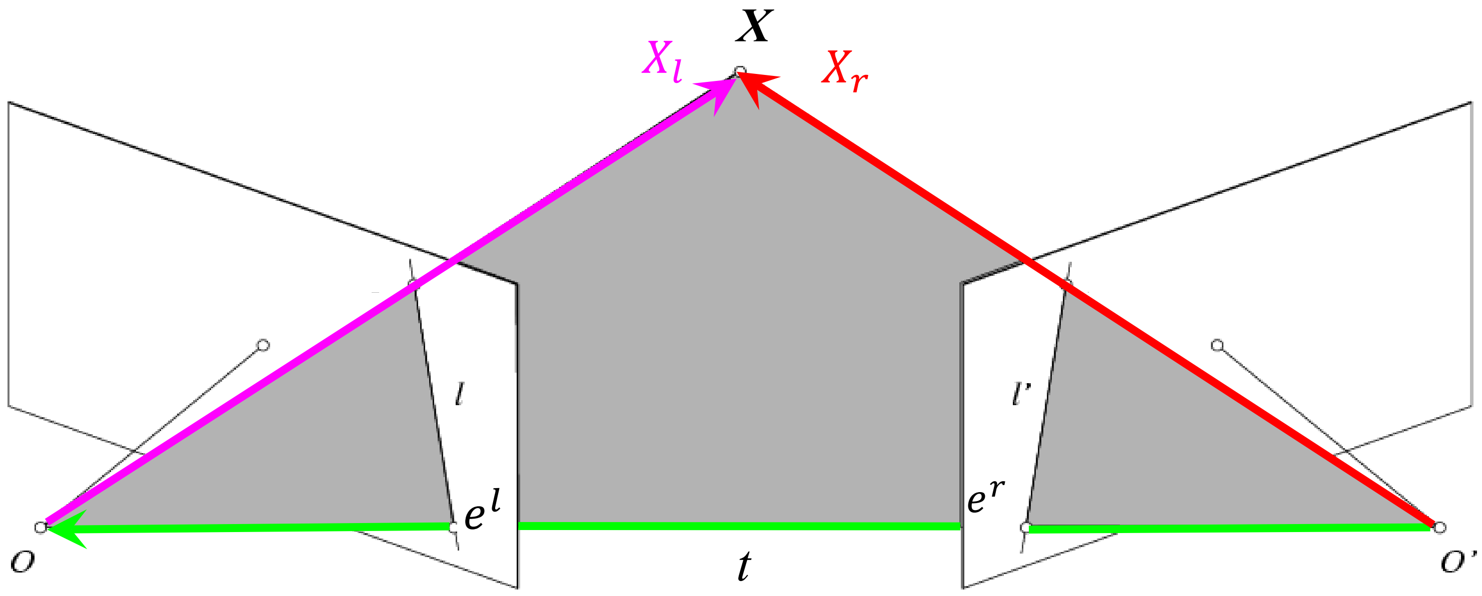
Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

Let E be $R[t]_\times$, $\quad X_l^T E X_r = X_l{}^T (R[t]_\times X_r) = (X_r - t)^T R^T R[t]_\times X_r$

Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

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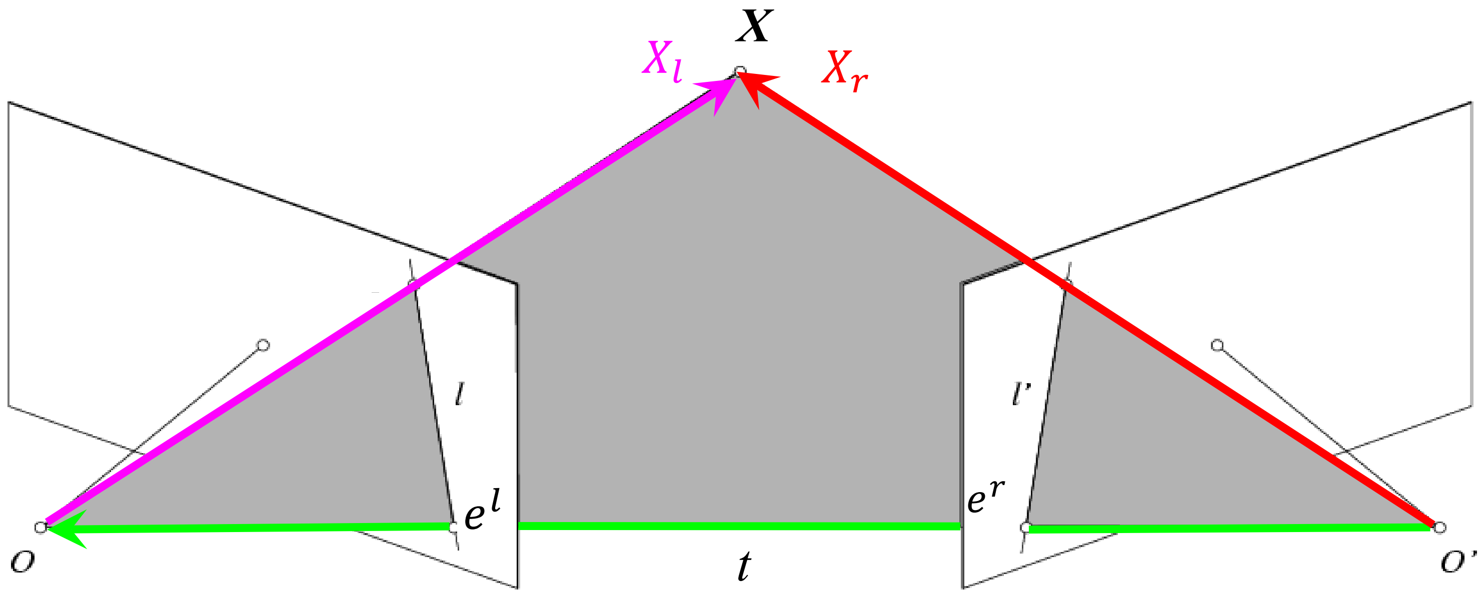
$$= (X_r - t) \cdot (t \times X_r)$$

Essential matrix (Longuet-Higgins, 1981)



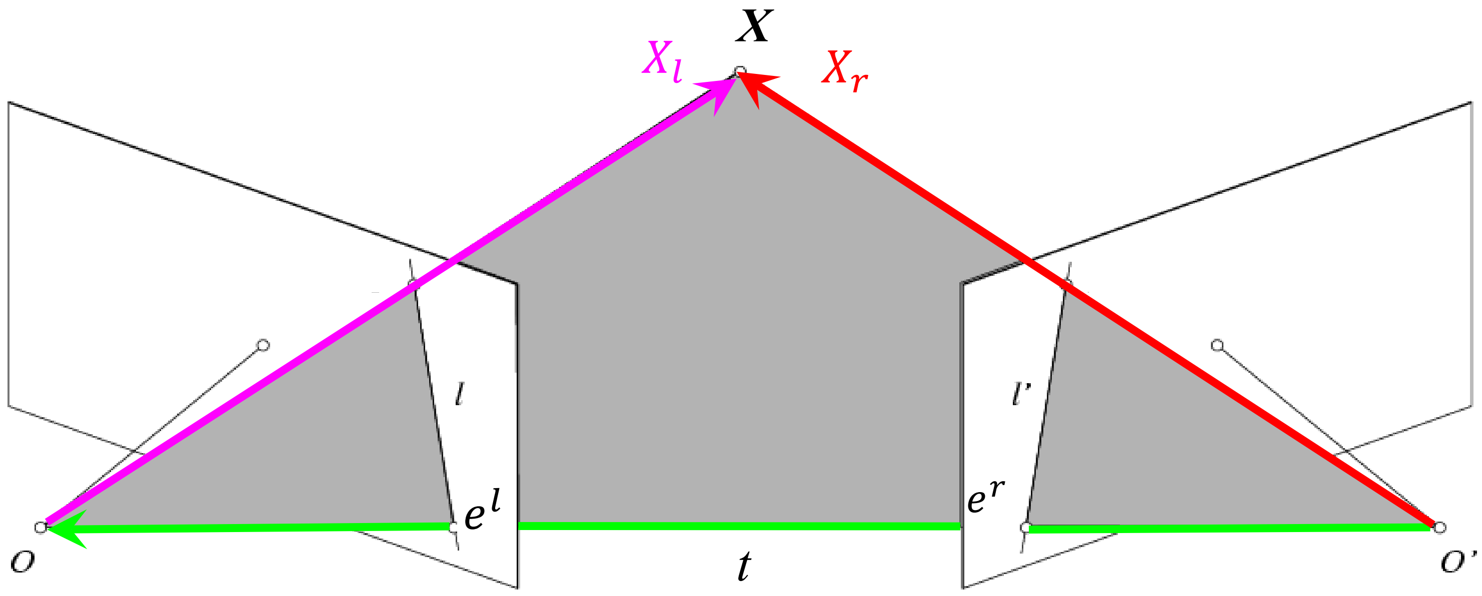
Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

Let E be $R[t]_\times$, $X_l^T E X_r = X_l{}^T (R[t]_\times X_r) = (X_r - t)^T R^T R[t]_\times X_r$

$$= (X_r - t) \cdot (t \times X_r)$$

$$= X_r \cdot (t \times X_r) - t \cdot (t \times X_r)$$

Essential matrix (Longuet-Higgins, 1981)



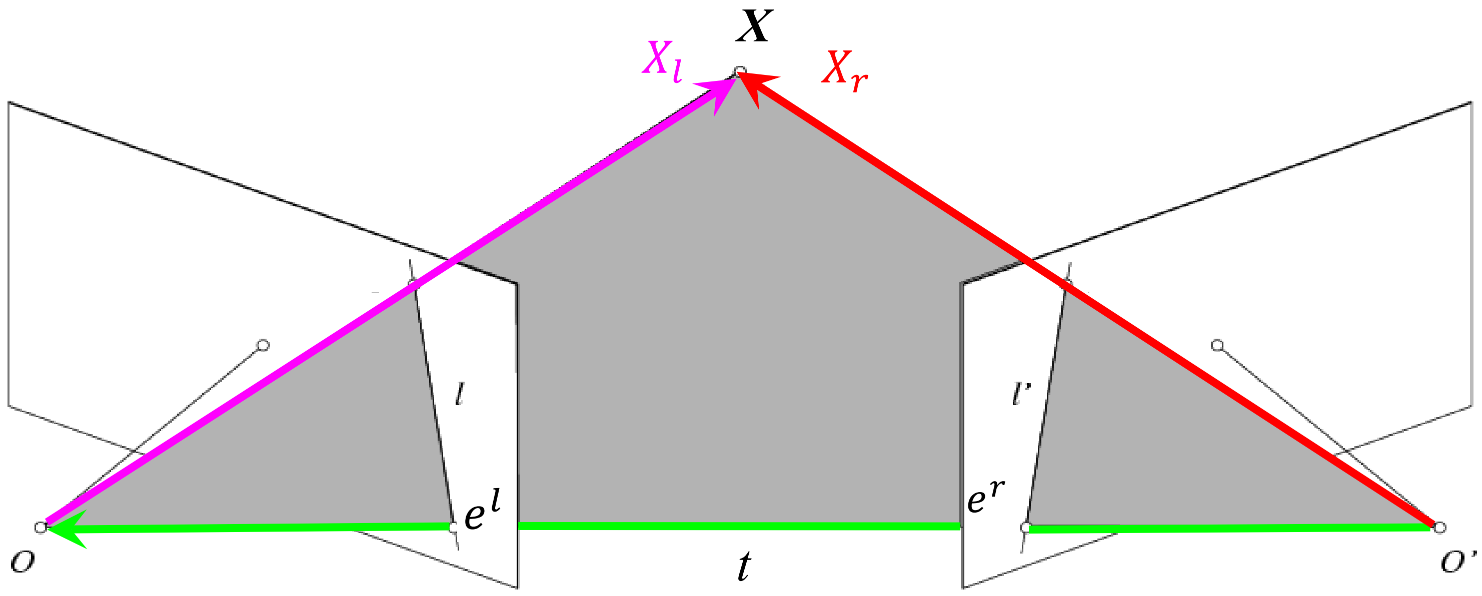
Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

Let E be $R[t]_\times$, $X_l^T E X_r = X_l^T (R[t]_\times X_r) = (X_r - t)^T R^T R[t]_\times X_r$

$$= (X_r - t) \cdot (t \times X_r)$$

$$= X_r \cdot (t \times X_r) - t \cdot (t \times X_r) = 0$$

Essential matrix (Longuet-Higgins, 1981)



Let R be rotation from right to left camera, then $X_l = R(X_r - t)$

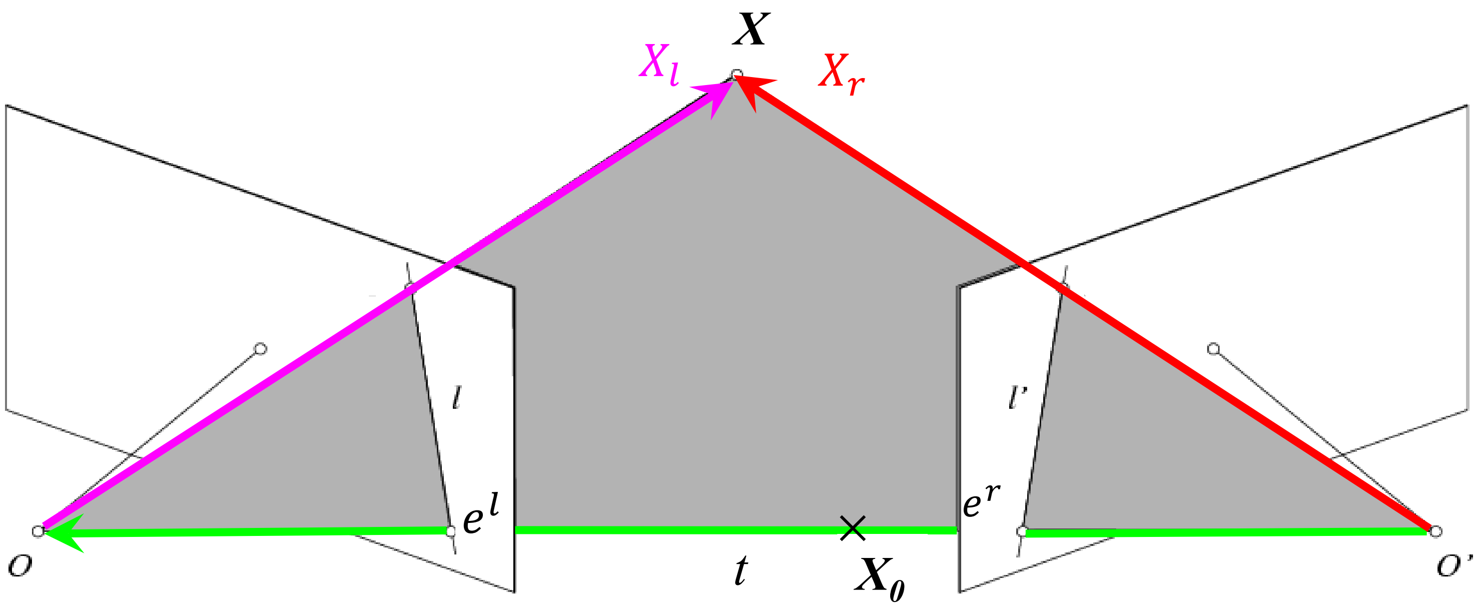
Let E be $R[t]_\times$, $X_l^T E X_r = X_l^T (R[t]_\times X_r) = (X_r - t)^T R^T R[t]_\times X_r$

$$= (X_r - t) \cdot (t \times X_r)$$

$$= X_r \cdot (t \times X_r) - t \cdot (t \times X_r) = 0$$

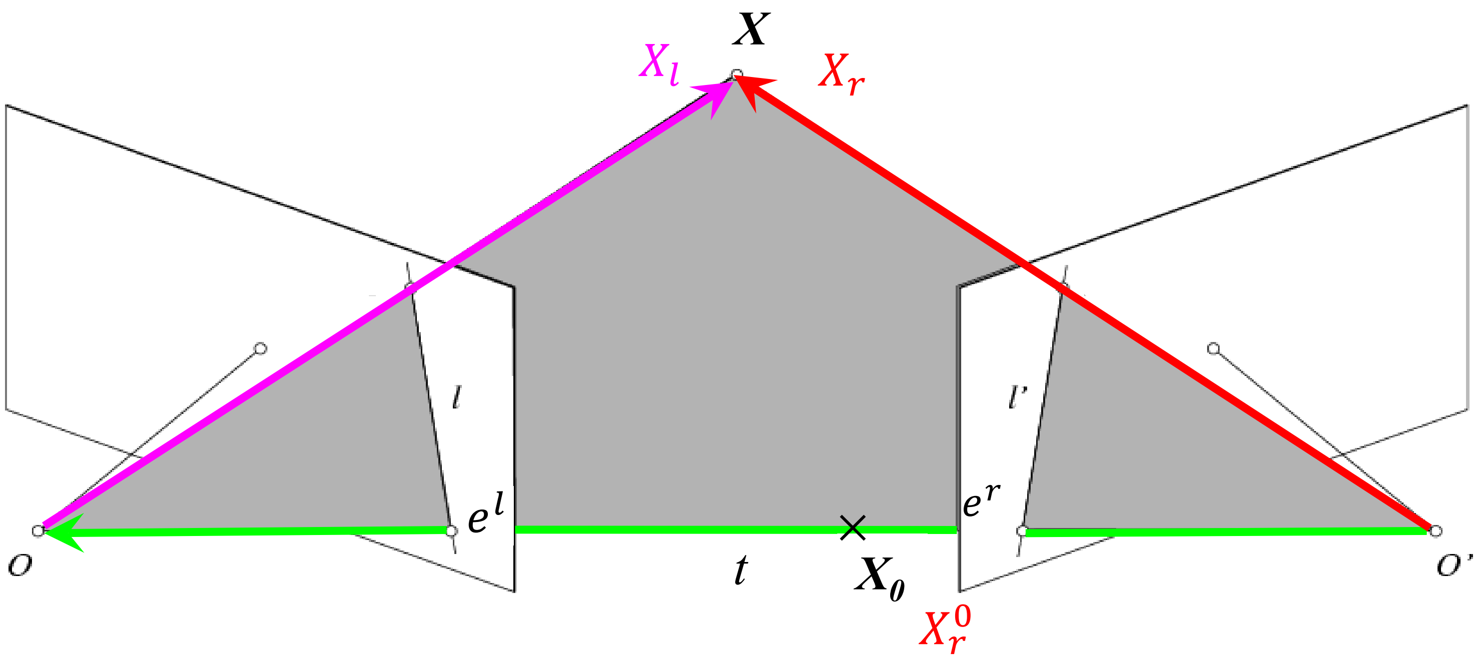
Note that $\text{rank}(E) \leq \text{rank}([t]_\times) \leq 2$

Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

Essential matrix (Longuet-Higgins, 1981)



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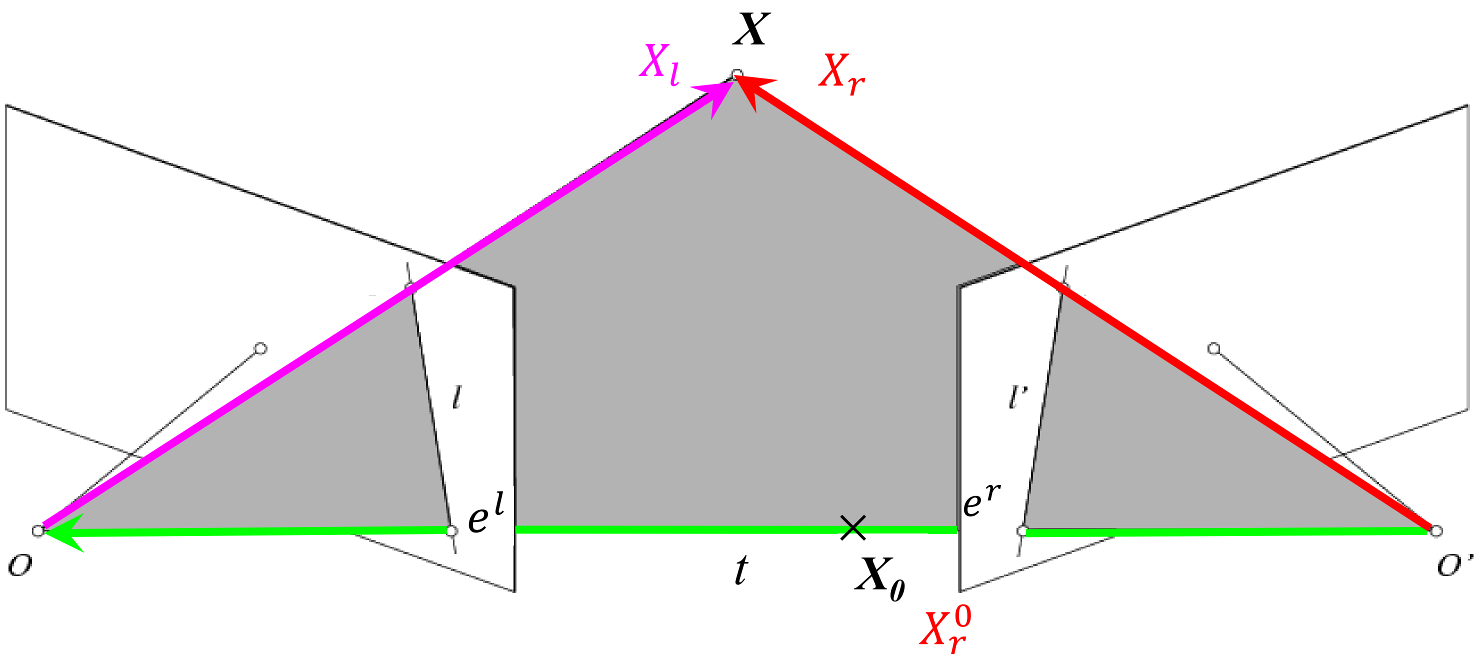
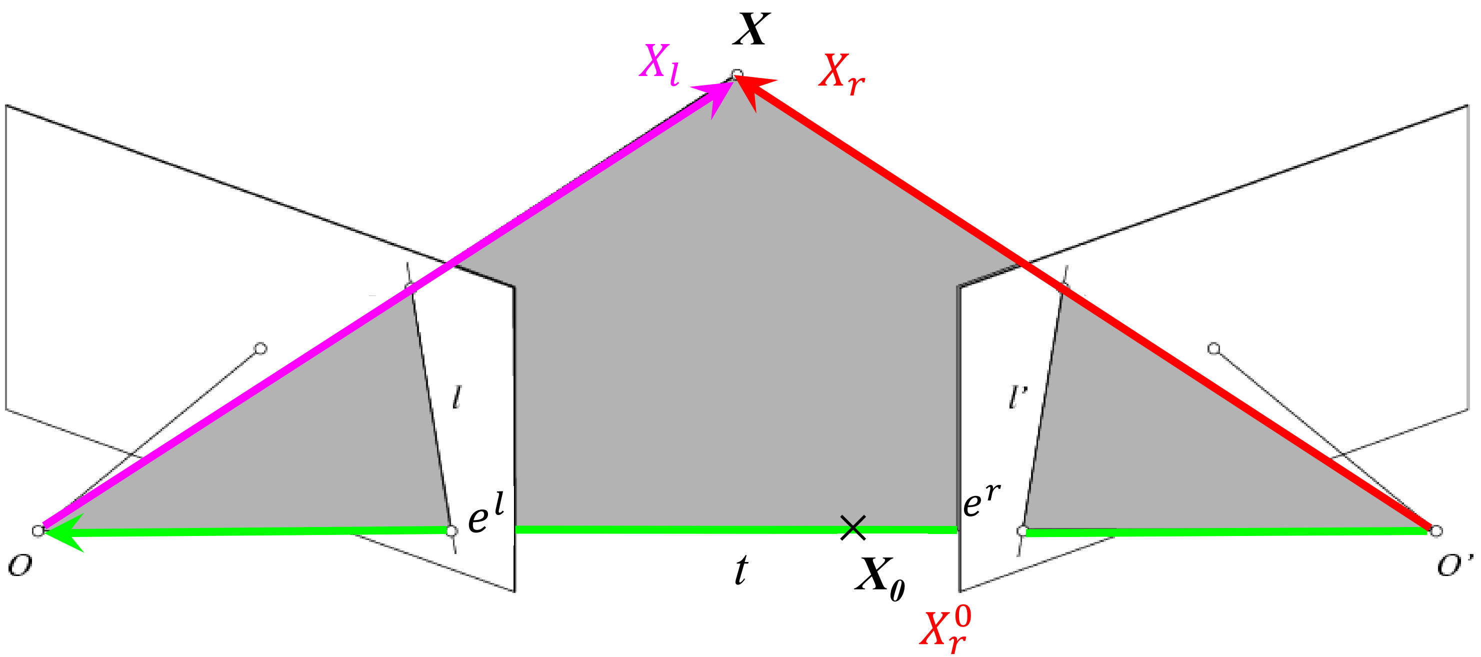
Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

$$EX_r^0 = (R[t]_\times)X_r^0$$

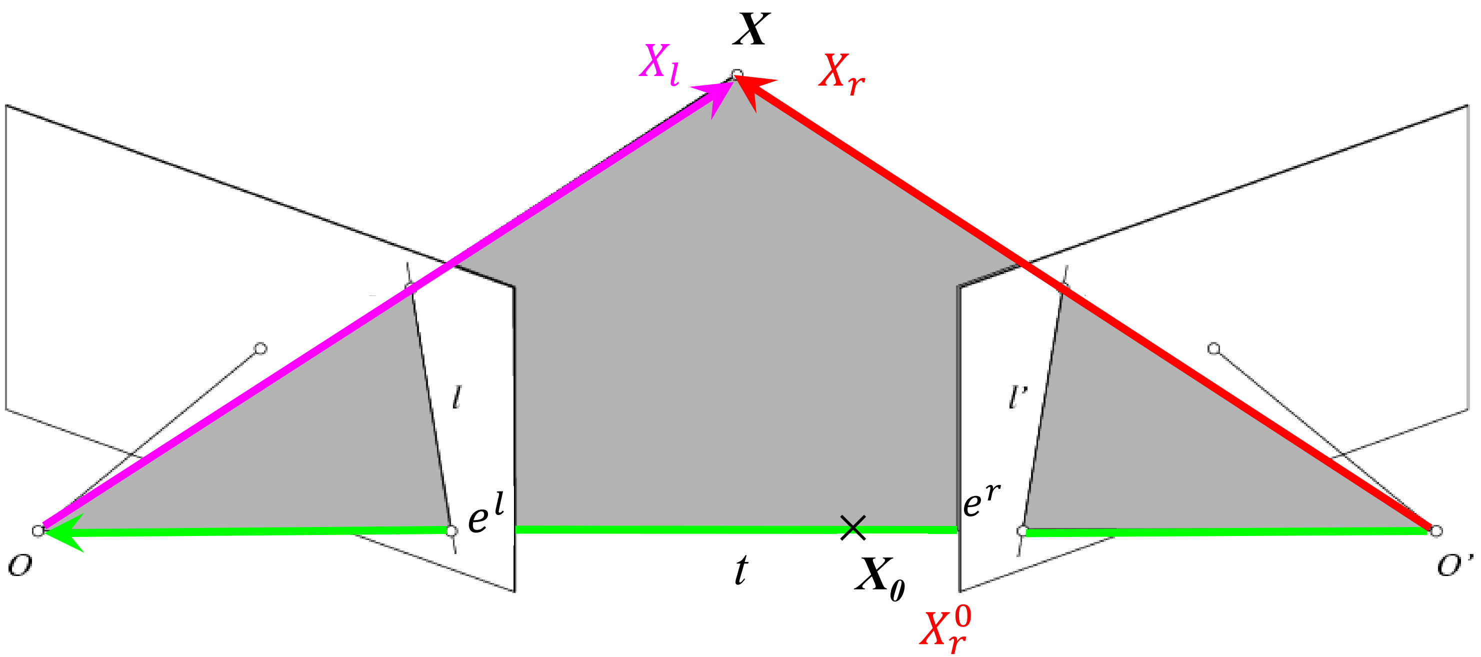
Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

$$EX_r^0 = (R[t]_\times)X_r^0 = R(t \times X_r^0)$$

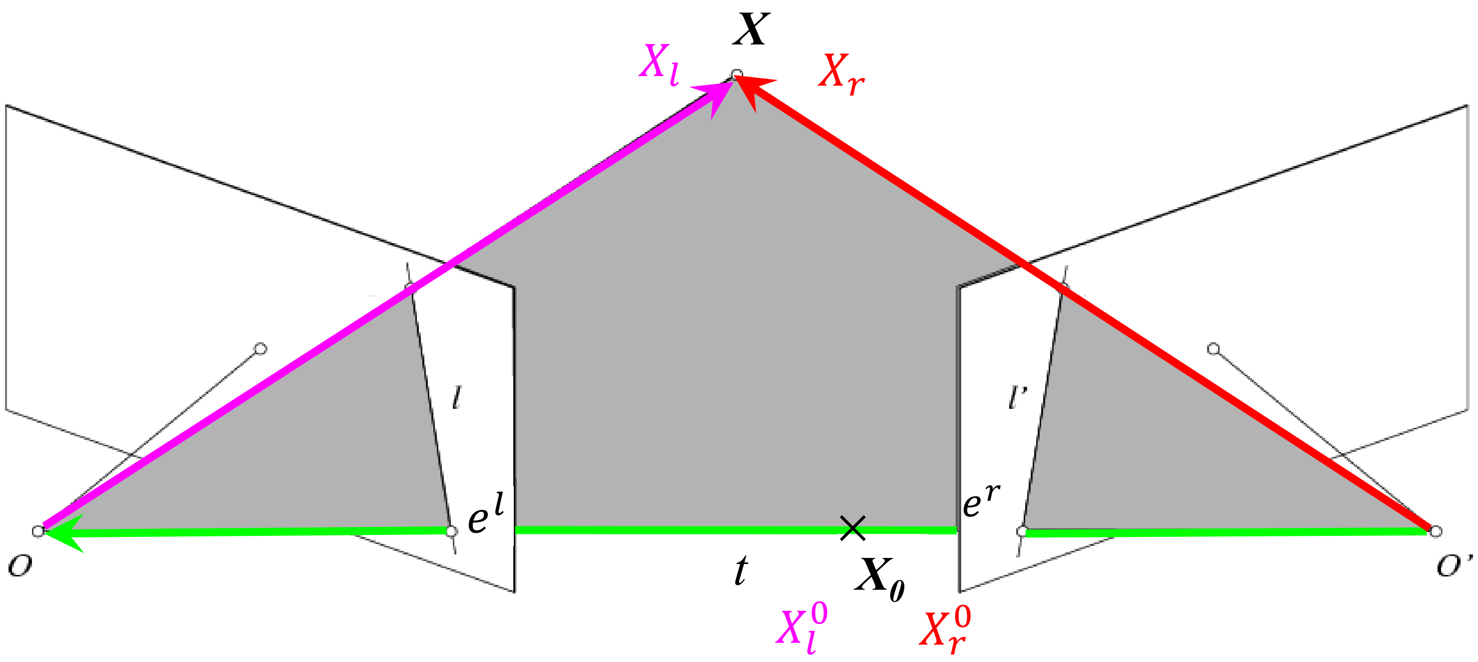
Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

$$E X_r^0 = (R[t]_\times) X_r^0 = R(t \times X_r^0) = 0$$

Essential matrix (Longuet-Higgins, 1981)

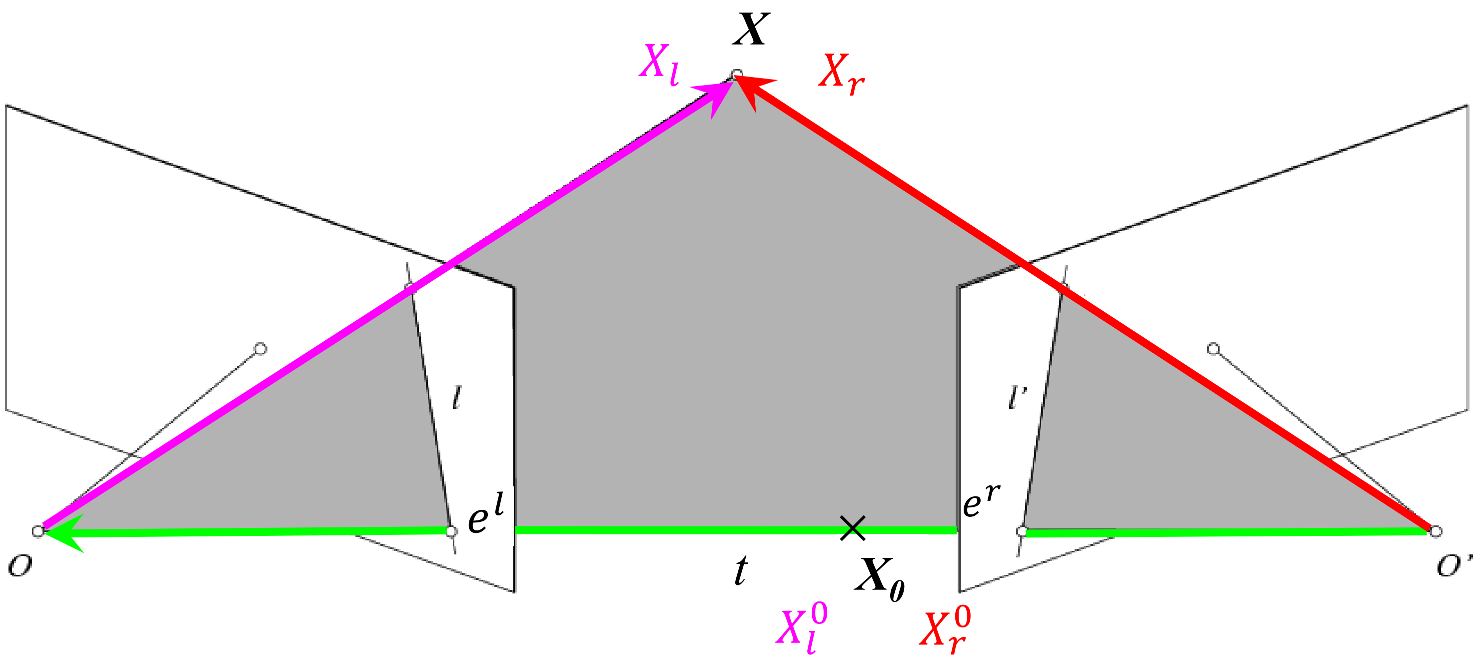


Consider any X_0 lies between the two centers of projection

$$EX_r^0 = (R[t]_\times)X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t)$$

Essential matrix (Longuet-Higgins, 1981)

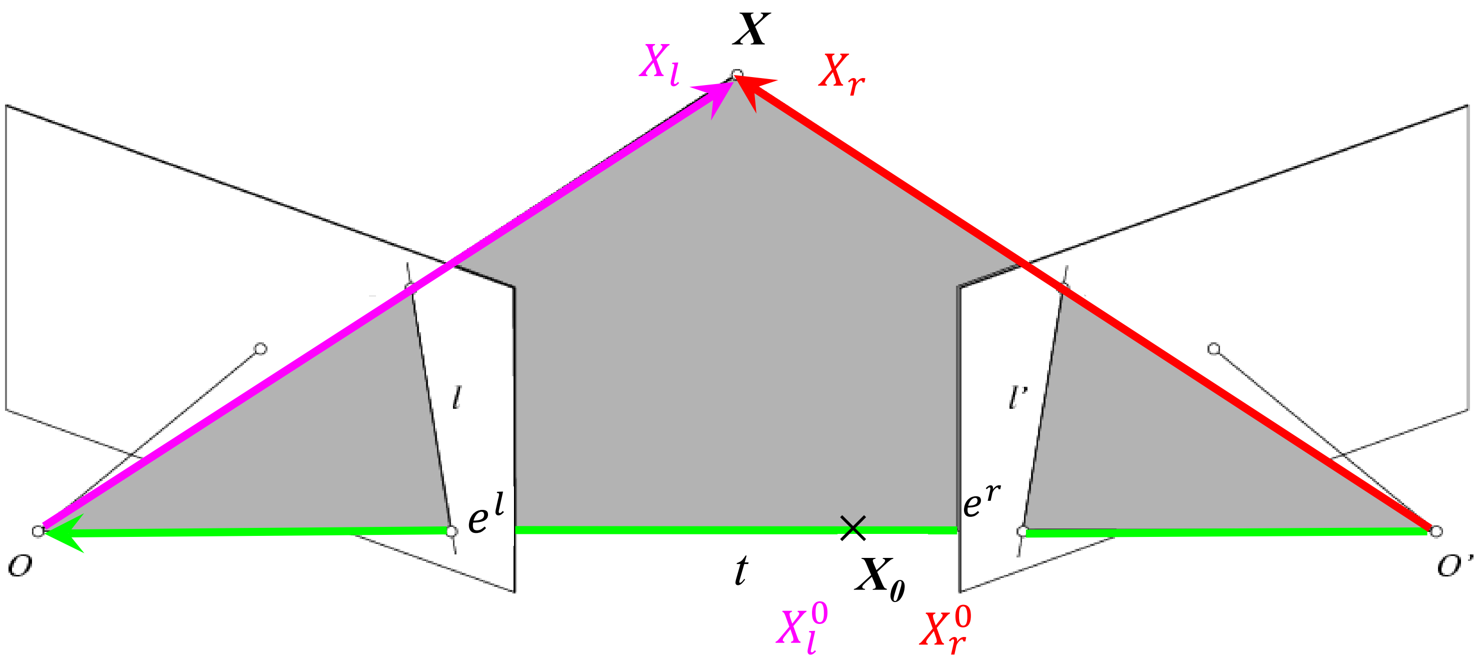


Consider any X_0 lies between the two centers of projection

$$EX_r^0 = (R[t]_\times)X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t) = ([t]_\times)^T R^T R(X_r^0 - t)$$

Essential matrix (Longuet-Higgins, 1981)



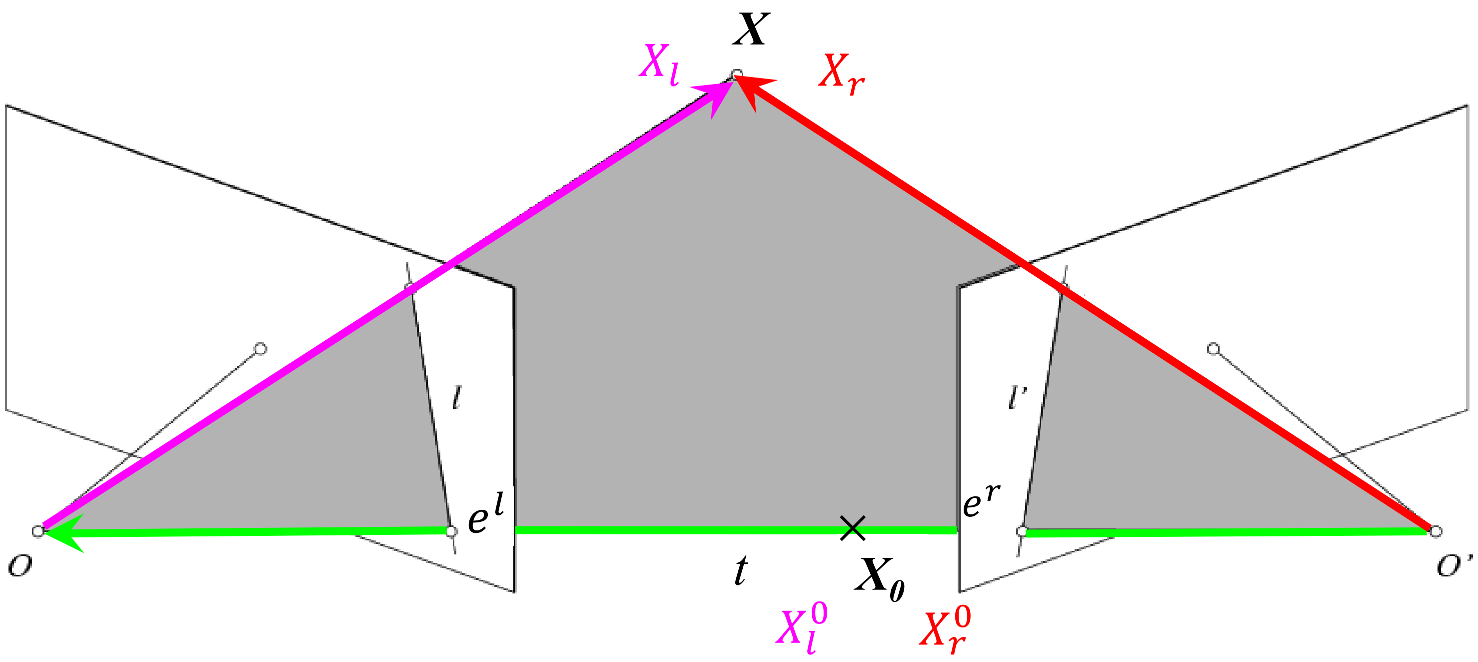
Consider any X_0 lies between the two centers of projection

$$EX_r^0 = (R[t]_\times)X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t) = ([t]_\times)^T R^T R(X_r^0 - t)$$

$$\underbrace{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}}_{[\mathbf{t}]_\times}$$

Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

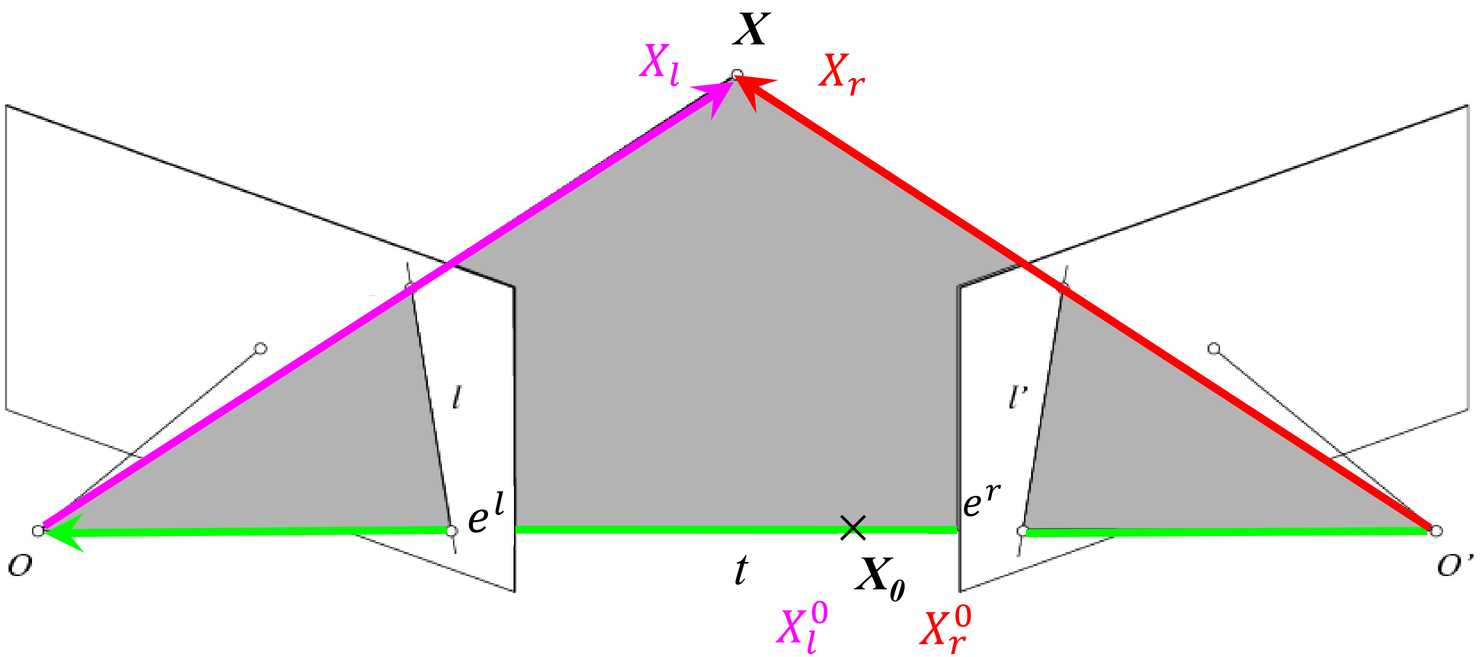
$$E X_r^0 = (R[t]_\times) X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t) = ([t]_\times)^T R^T R(X_r^0 - t)$$

$$= [-t]_\times (X_r^0 - t)$$

$$\underbrace{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}}_{[\mathbf{t}]_\times}$$

Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

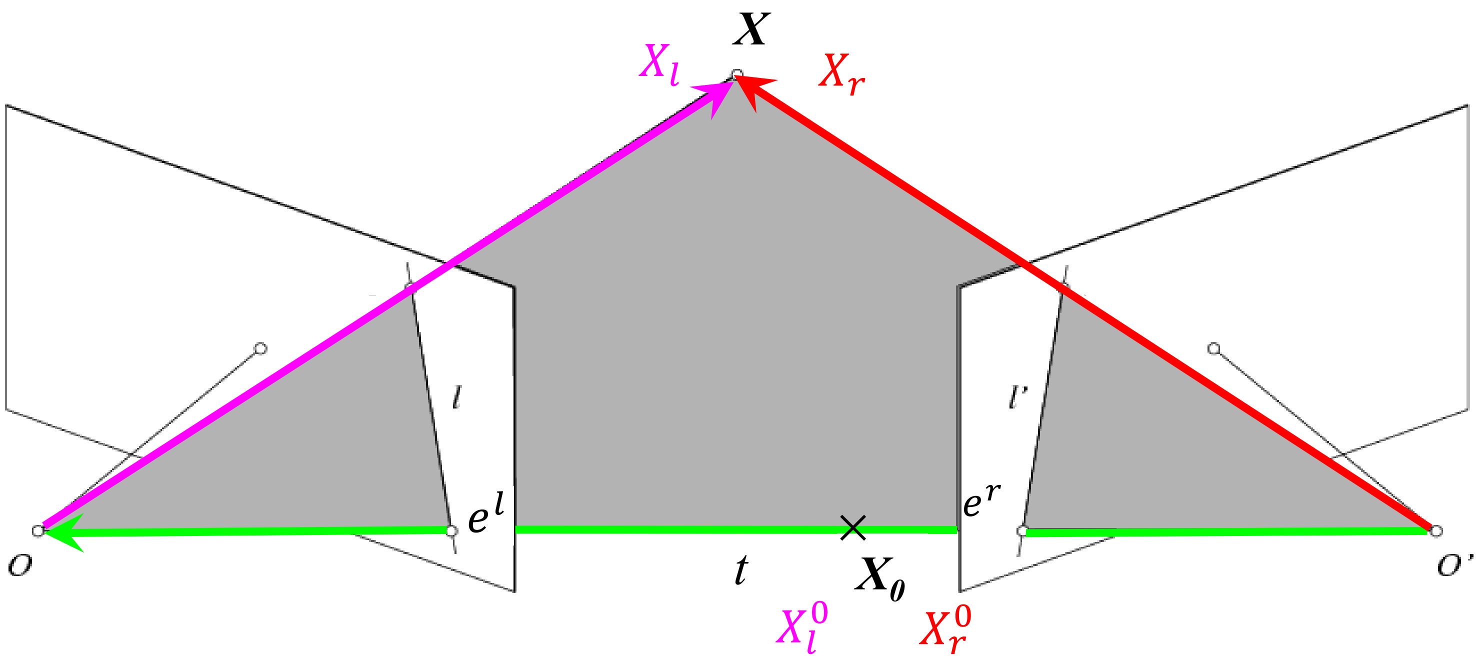
$$EX_r^0 = (R[t]_\times)X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t) = ([t]_\times)^T R^T R(X_r^0 - t)$$

$$= [-t]_\times (X_r^0 - t) = -t \times (X_r^0 - t)$$

$$\underbrace{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}}_{[\mathbf{t}]_\times}$$

Essential matrix (Longuet-Higgins, 1981)



Consider any X_0 lies between the two centers of projection

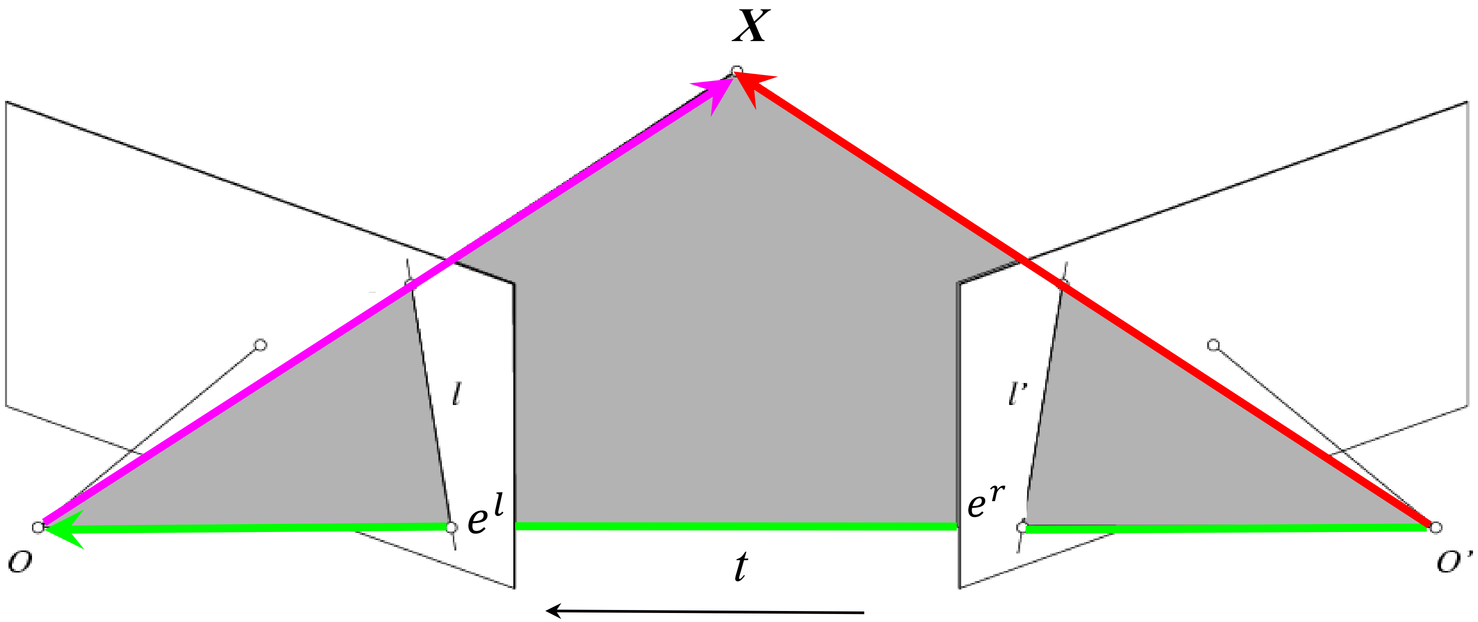
$$E X_r^0 = (R[t]_\times) X_r^0 = R(t \times X_r^0) = 0$$

$$E^T X_l^0 = (R[t]_\times)^T R(X_r^0 - t) = ([t]_\times)^T R^T R(X_r^0 - t)$$

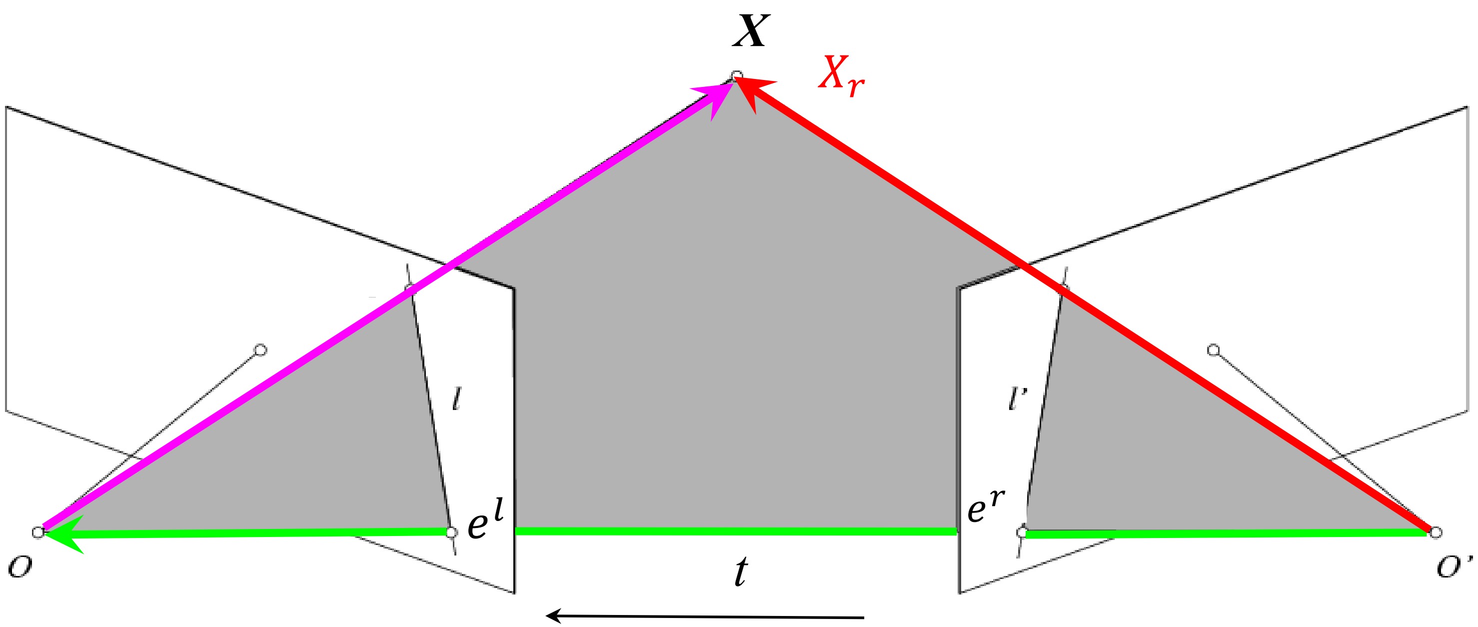
$$= [-t]_\times (X_r^0 - t) = -t \times (X_r^0 - t) = 0$$

$$\underbrace{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}}_{[\mathbf{t}]_\times}$$

Fundamental matrix (Faugeras and Luong, 1992)

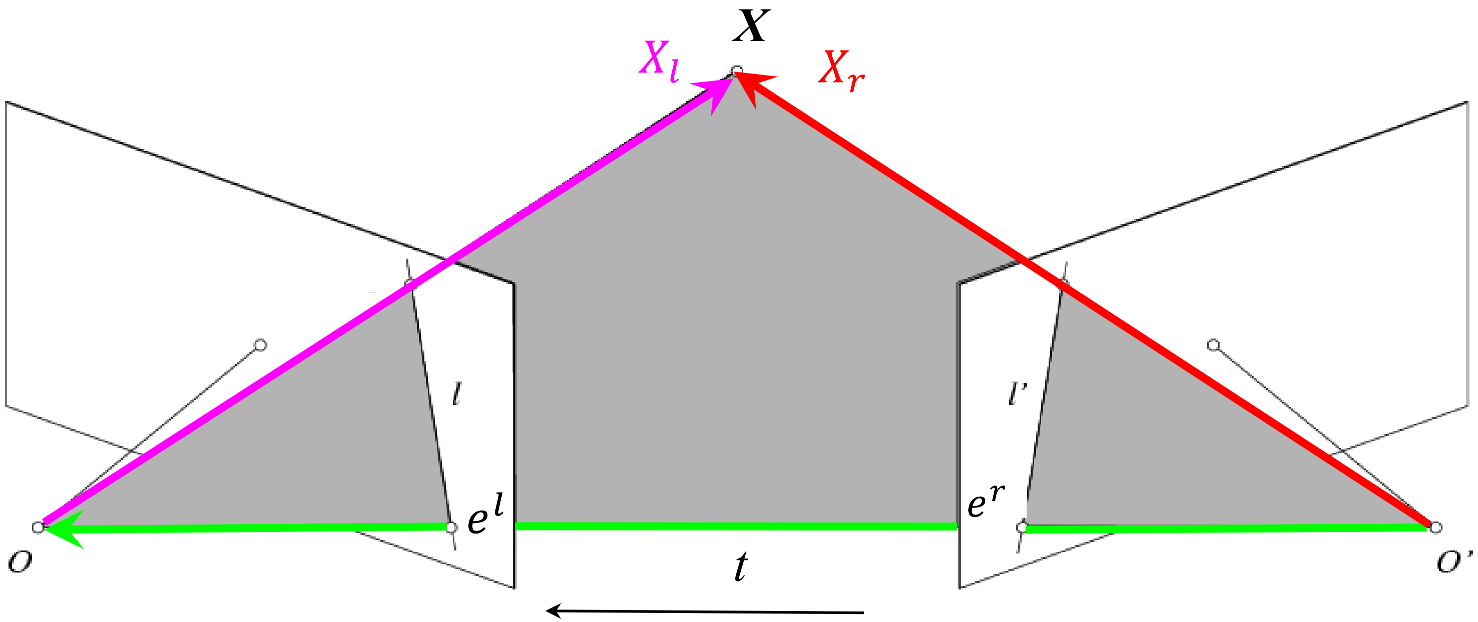


Fundamental matrix (Faugeras and Luong, 1992)



X_r: X in terms of Cartesian coordinate of right camera

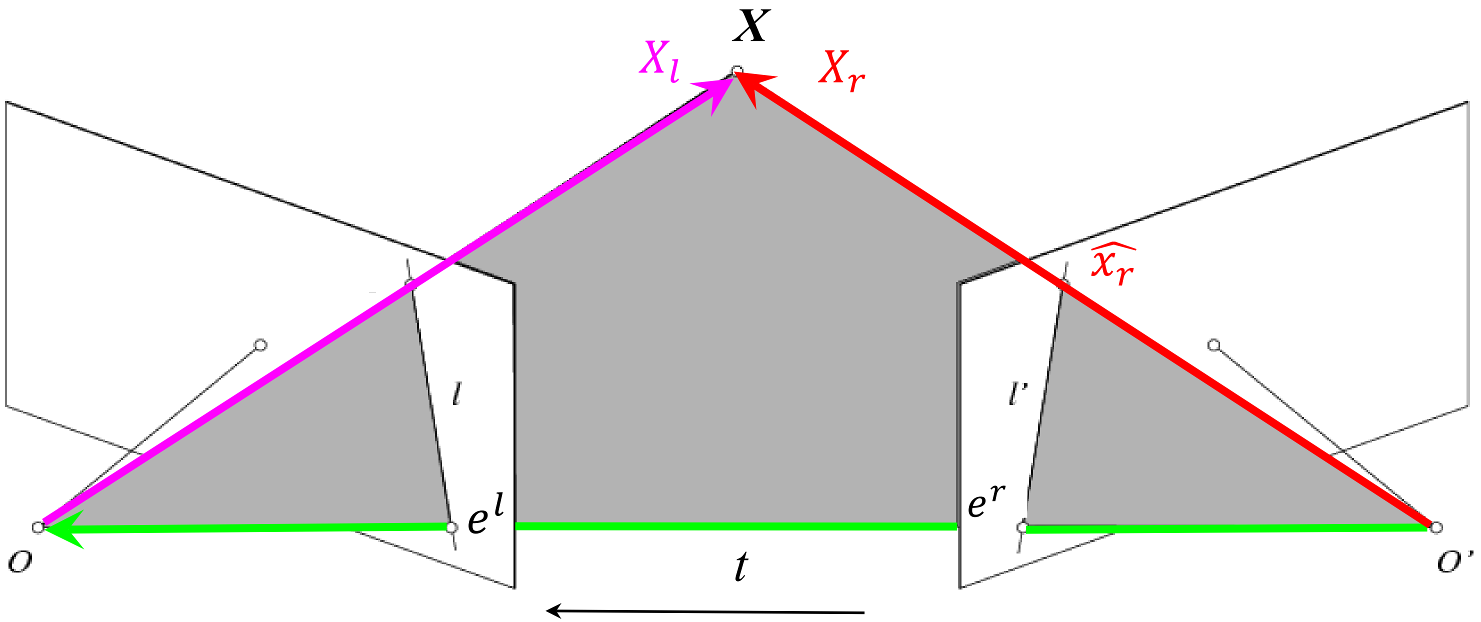
Fundamental matrix (Faugeras and Luong, 1992)



X_r: X in terms of Cartesian coordinate of right camera

X_l: X in terms of Cartesian coordinate of left camera

Fundamental matrix (Faugeras and Luong, 1992)

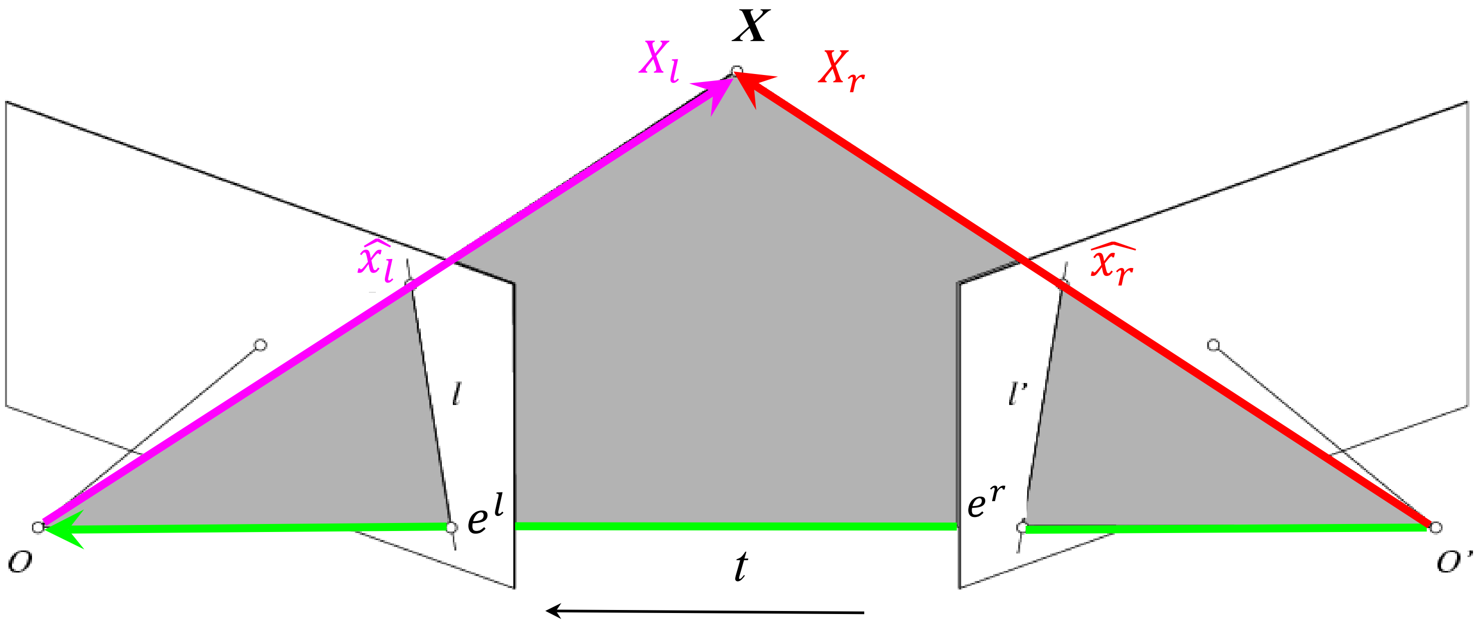


X_r: X in terms of Cartesian coordinate of right camera

X_l: X in terms of Cartesian coordinate of left camera

$\hat{x}_r$: X 's homogeneous coordinate of right view

Fundamental matrix (Faugeras and Luong, 1992)



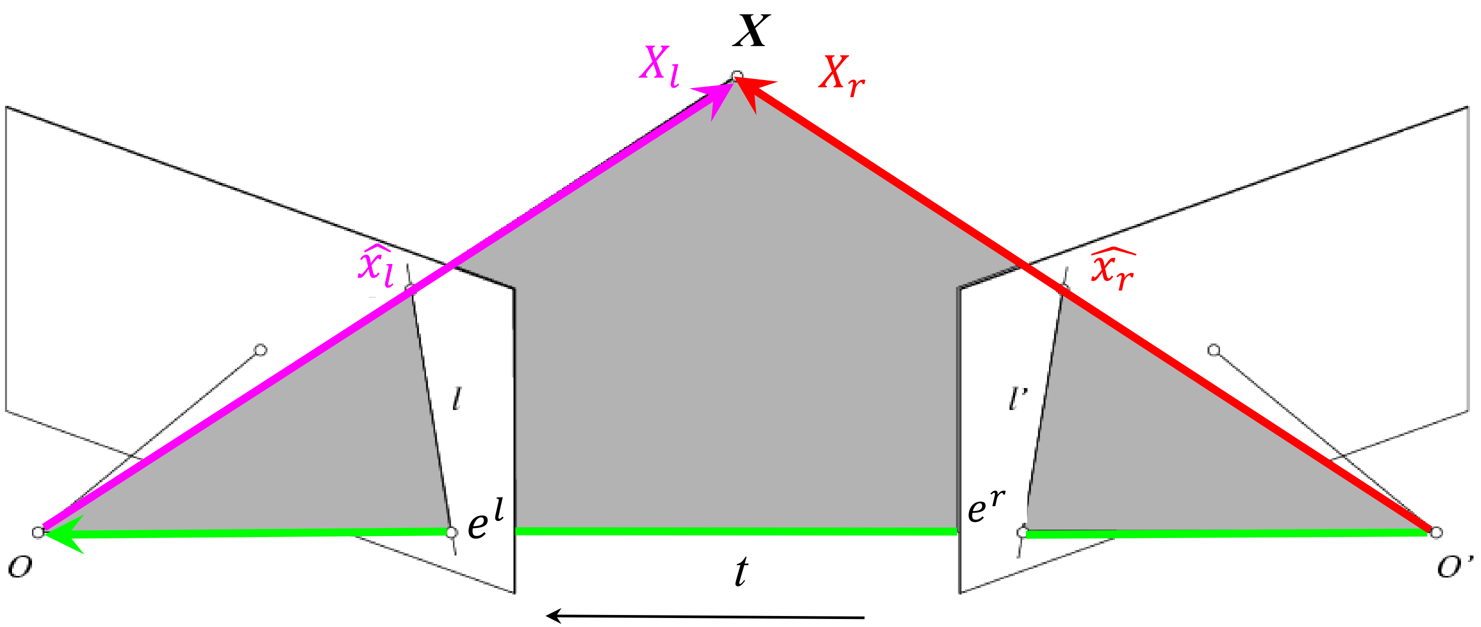
X_r: X in terms of Cartesian coordinate of right camera

X_l: X in terms of Cartesian coordinate of left camera

$\hat{x}_r$: X 's homogeneous coordinate of right view

$\hat{x}_l$: X 's homogeneous coordinate of left view

Fundamental matrix (Faugeras and Luong, 1992)



X_r: X in terms of Cartesian coordinate of right camera

X_l: X in terms of Cartesian coordinate of left camera

$\hat{x}_r$: X 's homogeneous coordinate of right view

$\hat{x}_l$: X 's homogeneous coordinate of left view

There exists F such that $\hat{x}_l^T F \hat{x}_r = 0$

Recall from last time

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$w\begin{bmatrix}U\\V\\1\end{bmatrix} = \begin{bmatrix}-d_i & s & U_0\\0 & -\alpha d_i & V_0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13} & t_x\\r_{21} & r_{22} & r_{23} & t_y\\r_{31} & r_{32} & r_{33} & t_z\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}$$

Recall from last time

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$w\begin{bmatrix} U \\ V \\ 1 \end{bmatrix} = \begin{bmatrix} -d_i & s & U_0 \\ 0 & -\alpha d_i & V_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Assume that $\mathbf{t} = \mathbf{0}$ (the origin of camera coordinate = COP)

and $\mathbf{R} = I$ (camera coordinate is perfectly aligned)

James Hays

Recall from last time

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$w\begin{bmatrix}U \\ V \\ 1\end{bmatrix} = \begin{bmatrix}-d_i & s & U_0 \\ 0 & -\alpha d_i & V_0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z\end{bmatrix}\begin{bmatrix}X \\ Y \\ Z \\ 1\end{bmatrix}$$

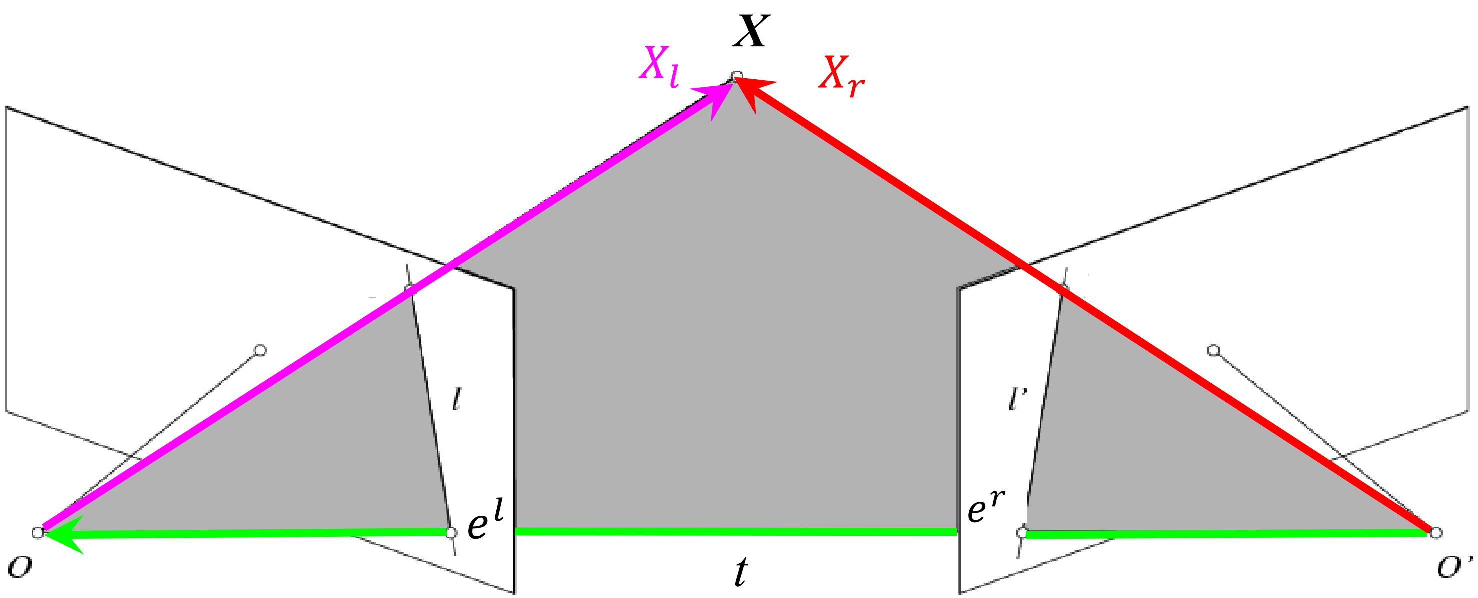
Assume that $\mathbf{t} = \mathbf{0}$ (the origin of camera coordinate = COP)

and $\mathbf{R} = I$ (camera coordinate is perfectly aligned)

$$\underbrace{\mathbf{x}}_{\text{2d projection location in homogenous coordinate}} = \mathbf{K} \underbrace{\mathbf{X}}_{\text{3d location in Cartesian coordinate}}$$

James Hays

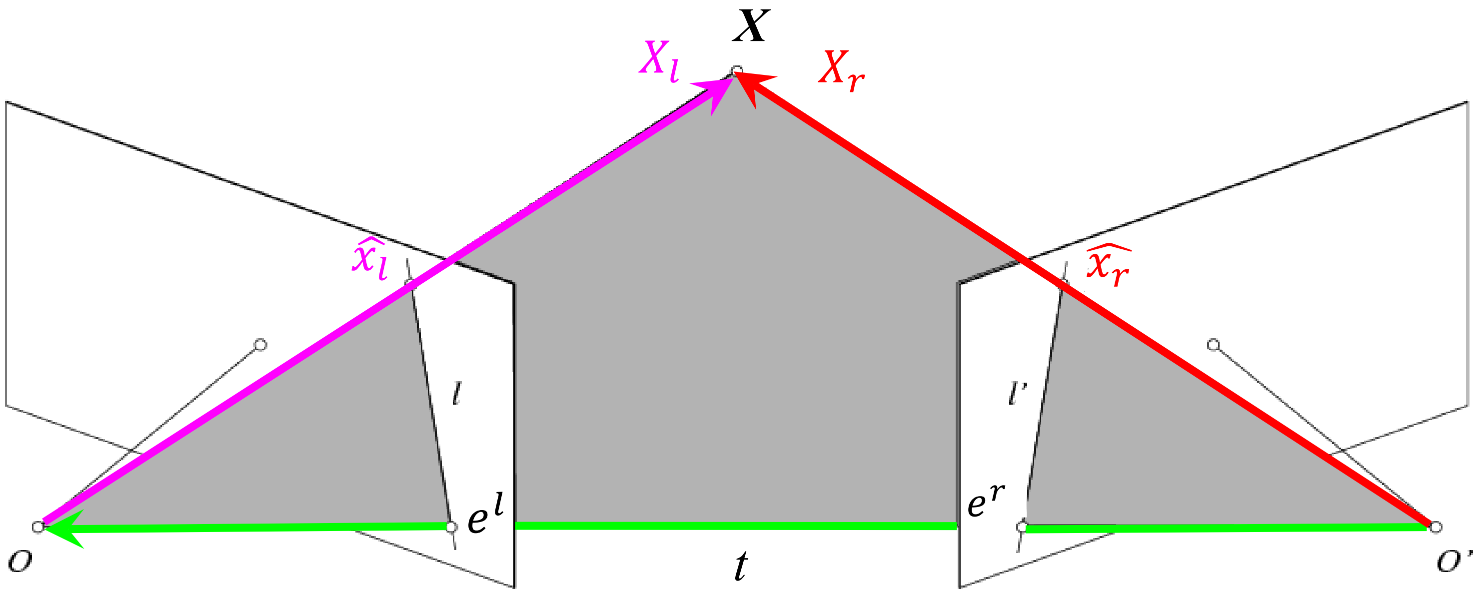
Fundamental matrix



X_l: X in terms of Cartesian coordinate of left camera

X_r: X in terms of Cartesian coordinate of right camera

Fundamental matrix



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$\hat{x}_l = K_l X_l$: X's homogeneous coordinate of left view
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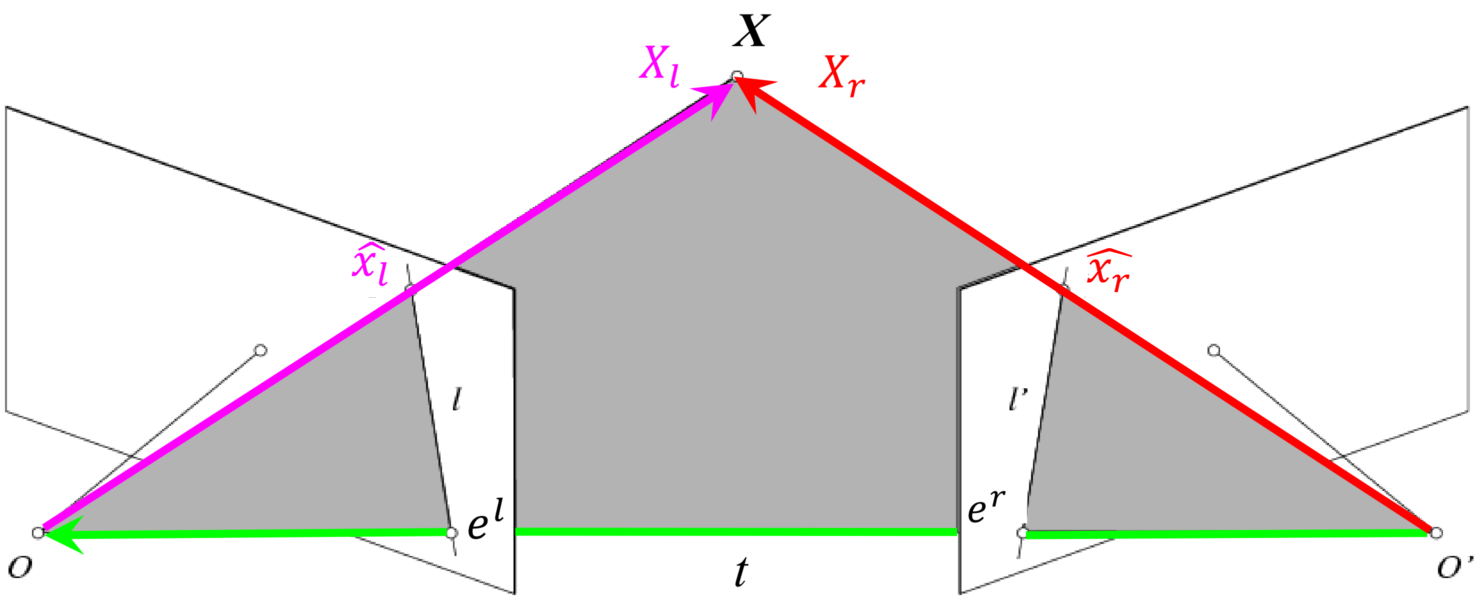
Fundamental matrix



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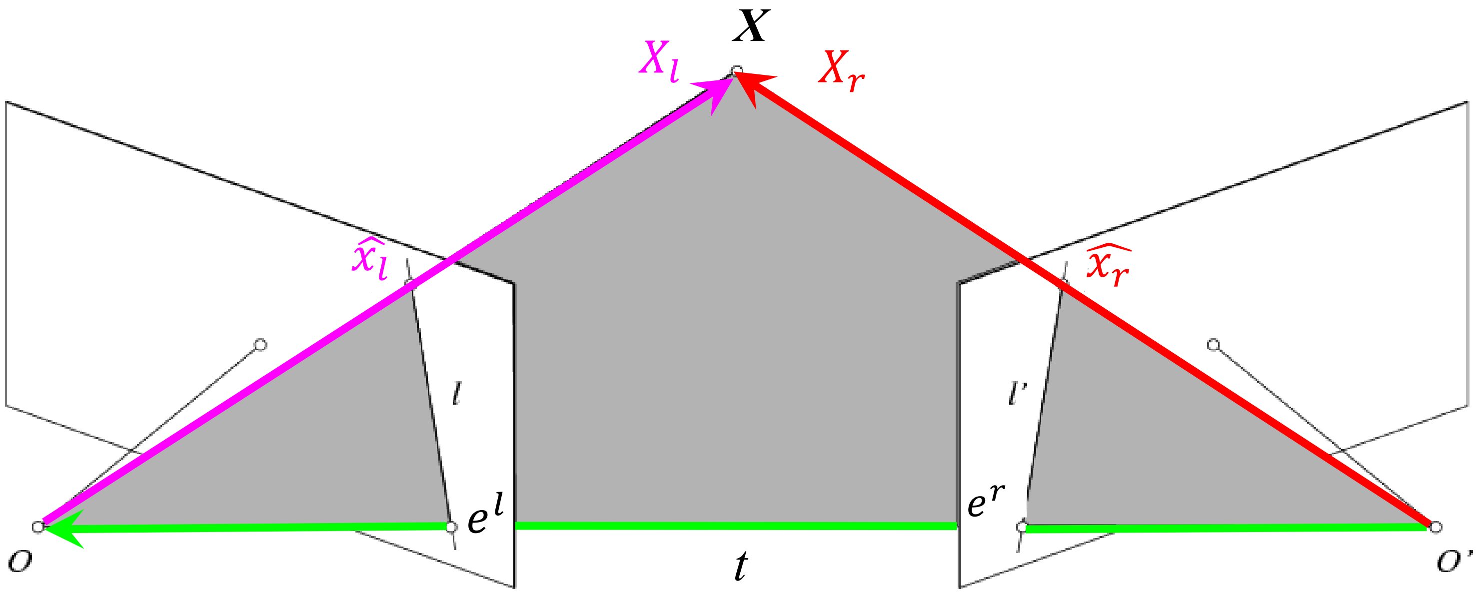
$$\hat{x}_l^{\,T} \underbrace{(K_l^T)^{-1} E K_r^{-1}}_{F} \hat{x}_r$$

Fundamental matrix



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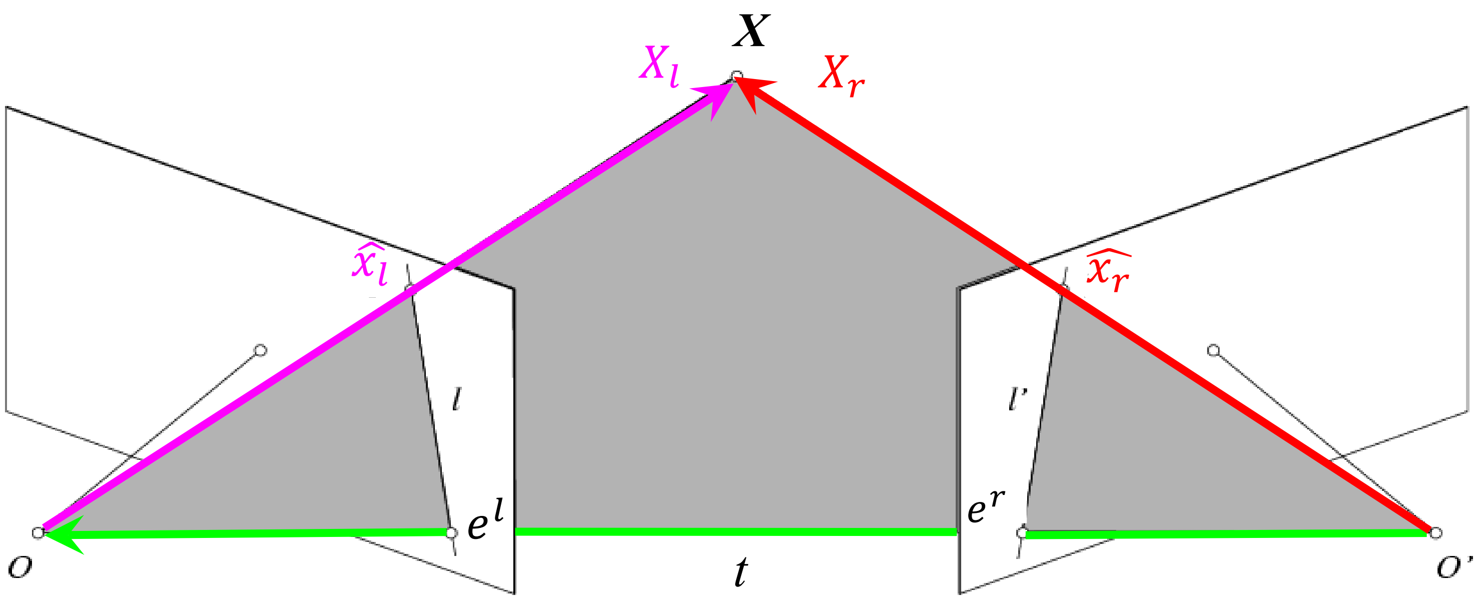

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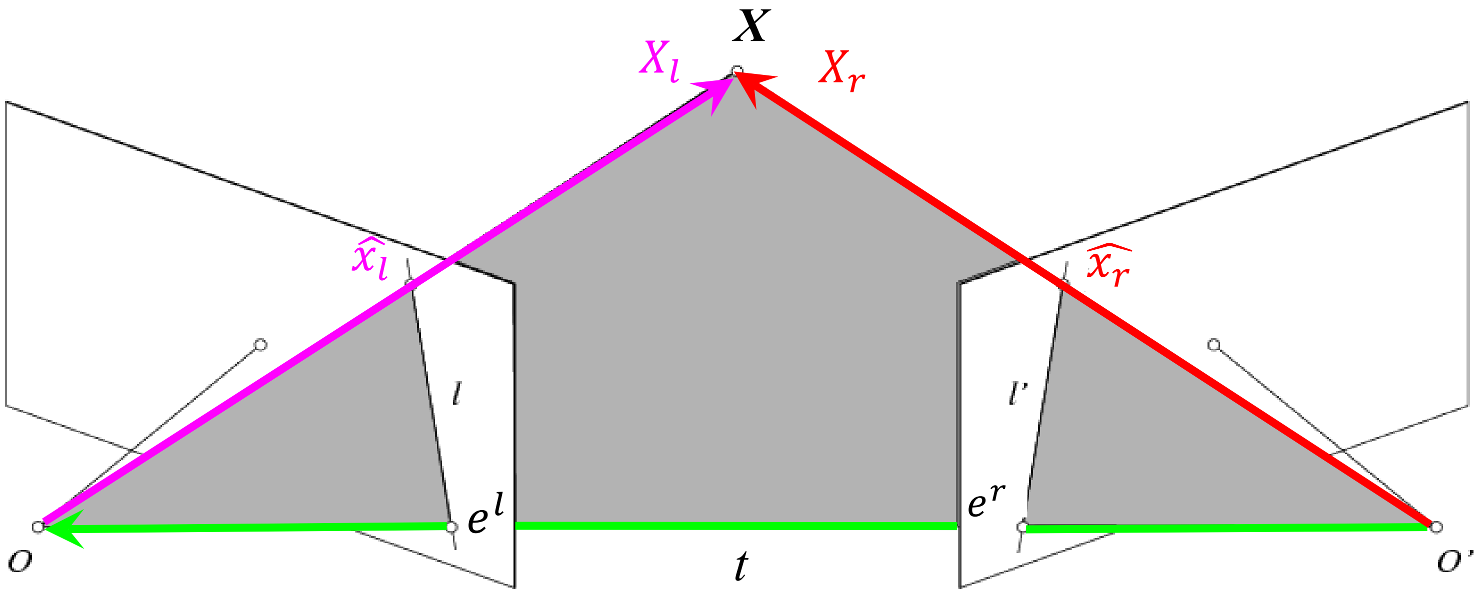

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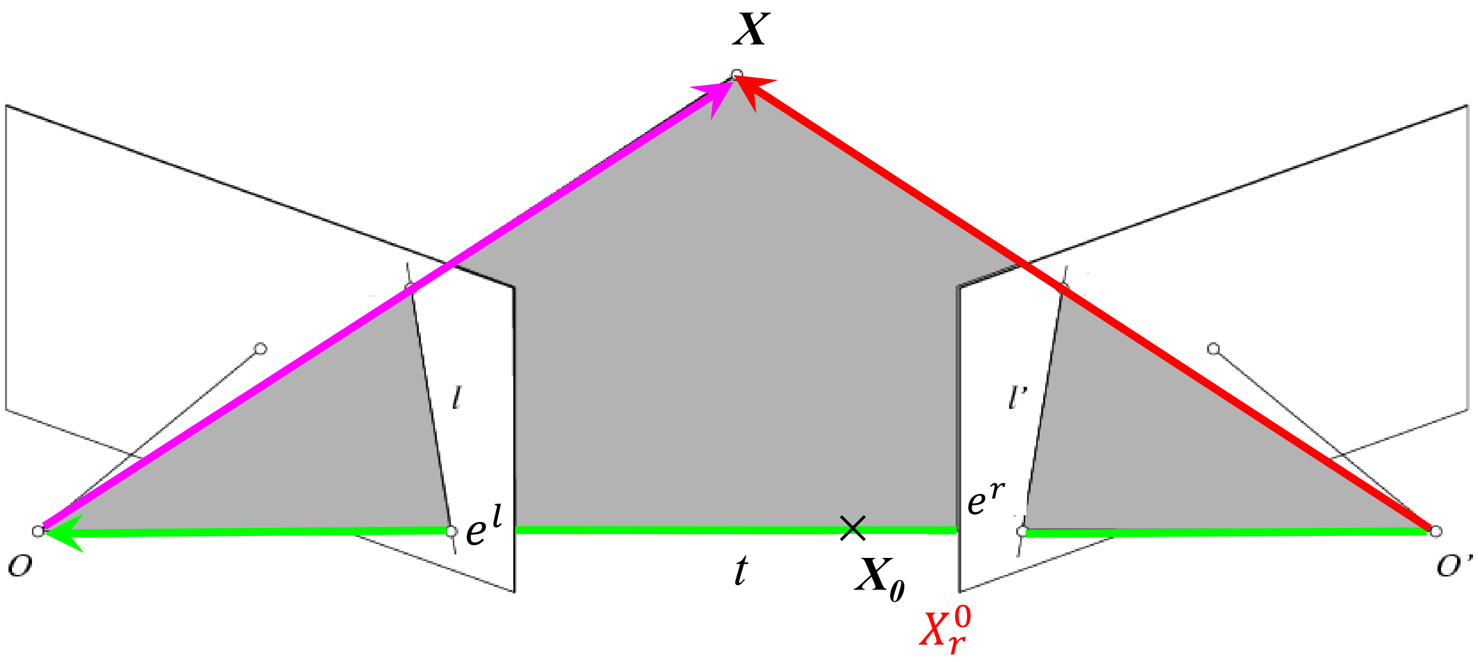
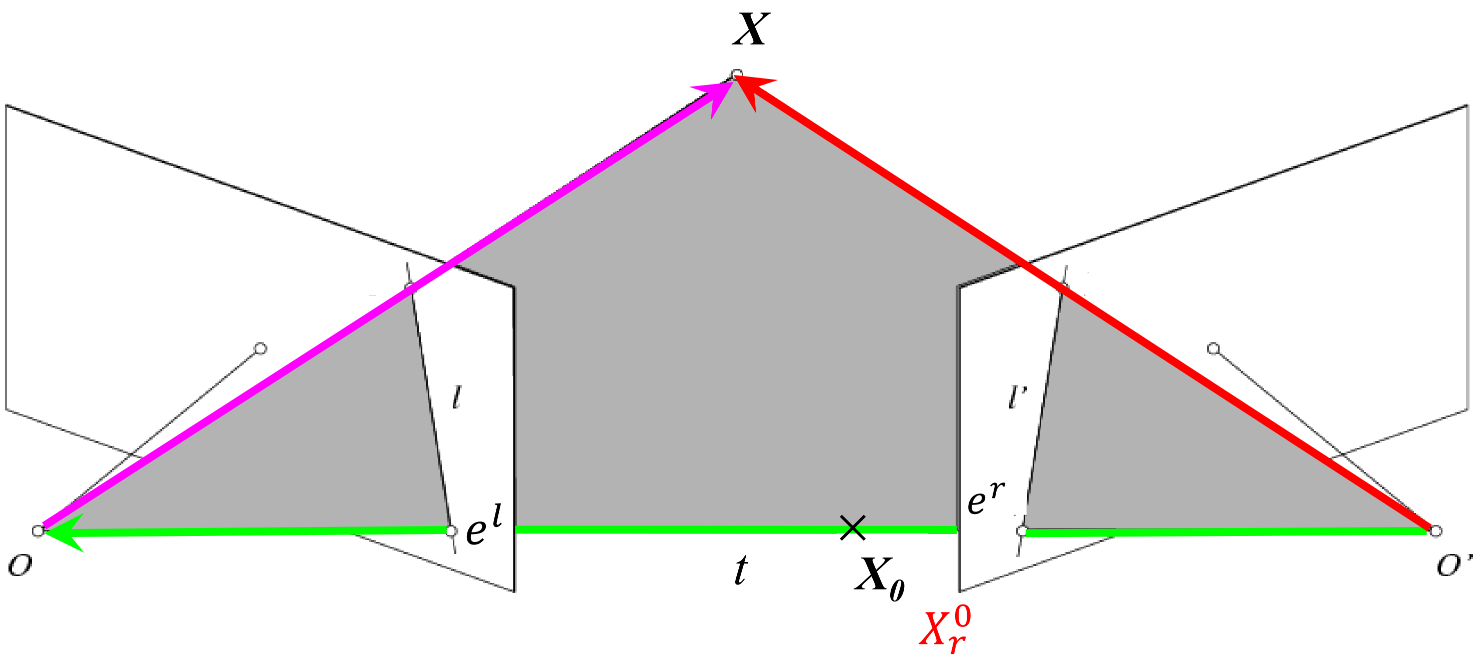
Epipoles and fundamental matrix



Consider any X_0 lie between the two centers of projection

Epipoles and fundamental matrix



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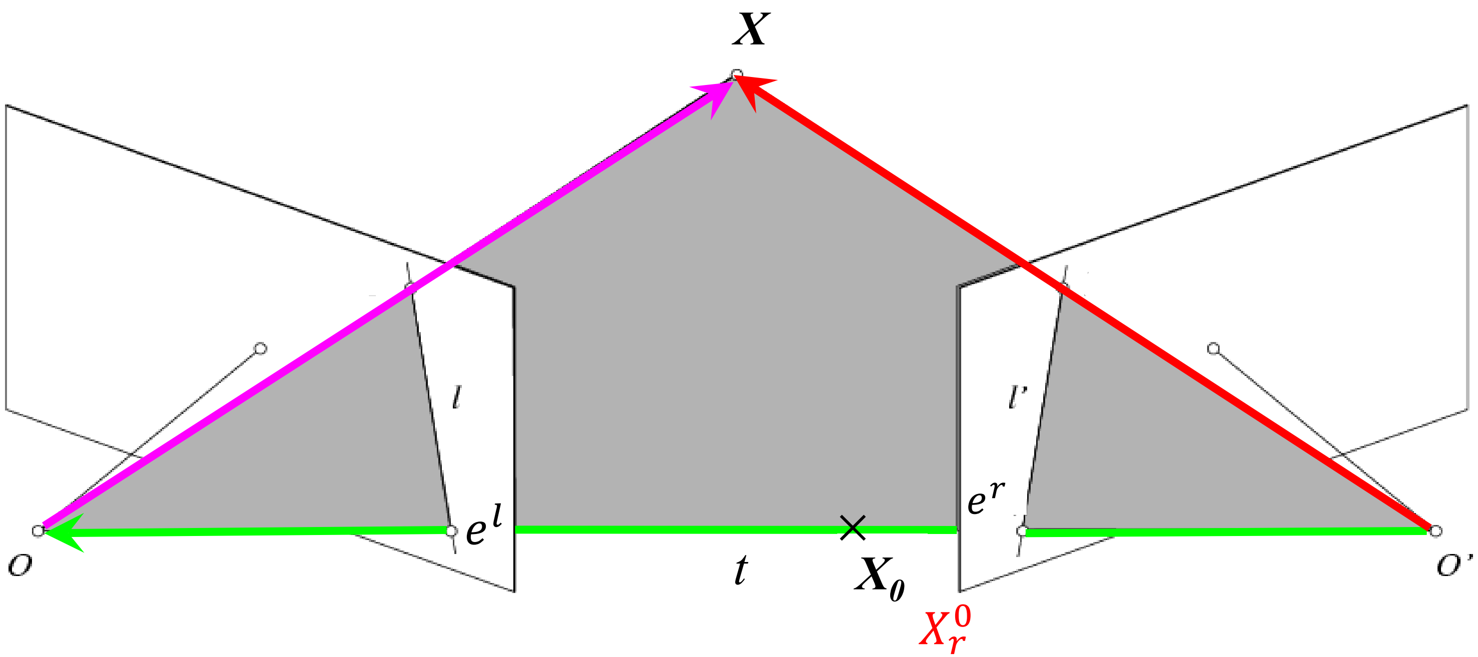
$$Fe^r = \underbrace{(K_l^T)^{-1}EK_r^{-1}}_{F}e^r = (K_l^T)^{-1}EK_r^{-1}\underbrace{K_rX_r^0}_{e^r} = (K_l^T)^{-1}EX_r^0$$

Epipoles and fundamental matrix



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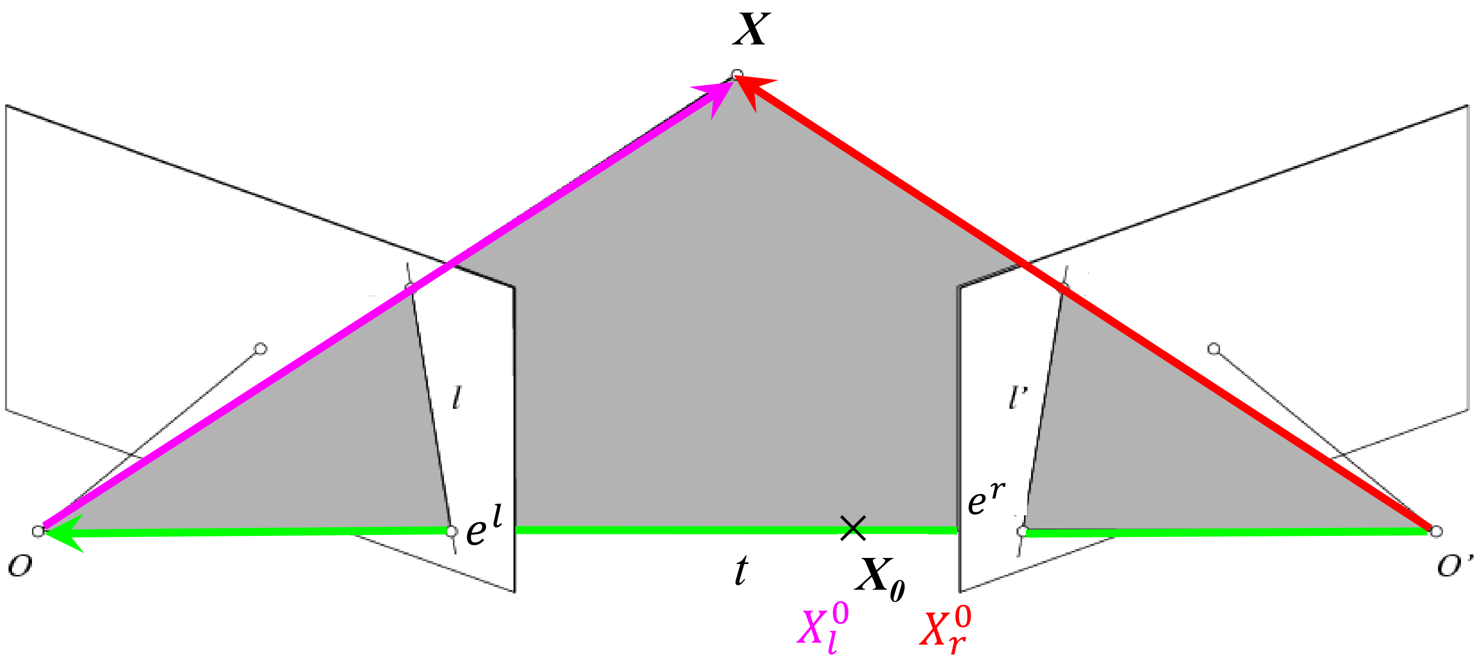

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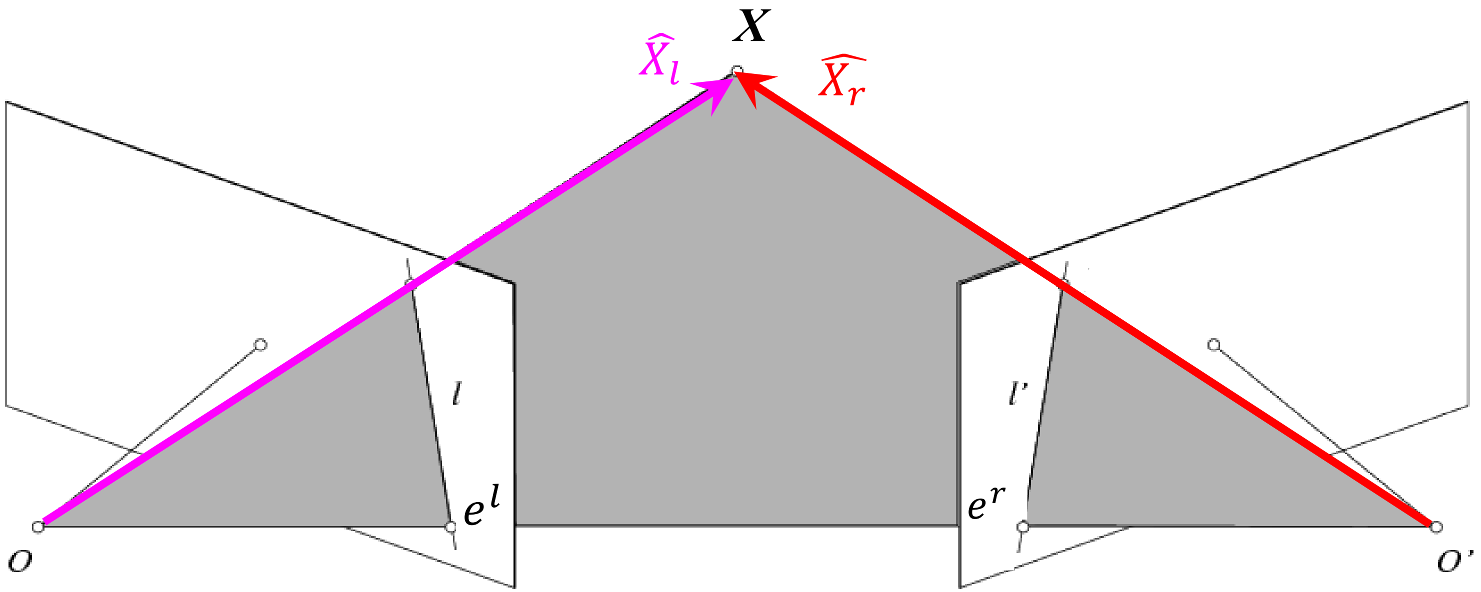
$$F^T e^l = \underbrace{(K_r^T)^{-1} E^T K_l^{-1}}_{F} e^l = (K_r^T)^{-1} E^T K_l^{-1} \underbrace{K_l X_l^0}_{e^l} = (K_r^T)^{-1} E^T X_l^0 = 0$$

Review: Essential matrix



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There exists E such that $\hat{X}_l^T \underbrace{E}_{R[t]_\times} \hat{X}_r = 0$

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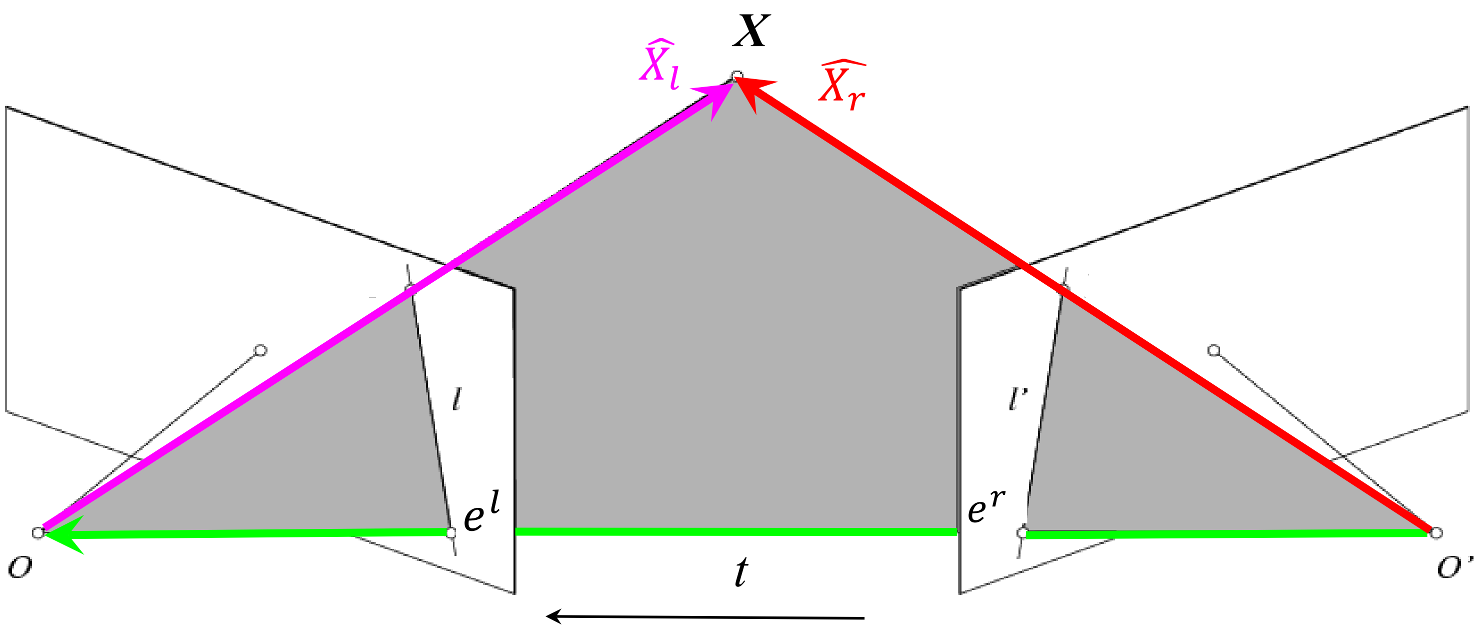

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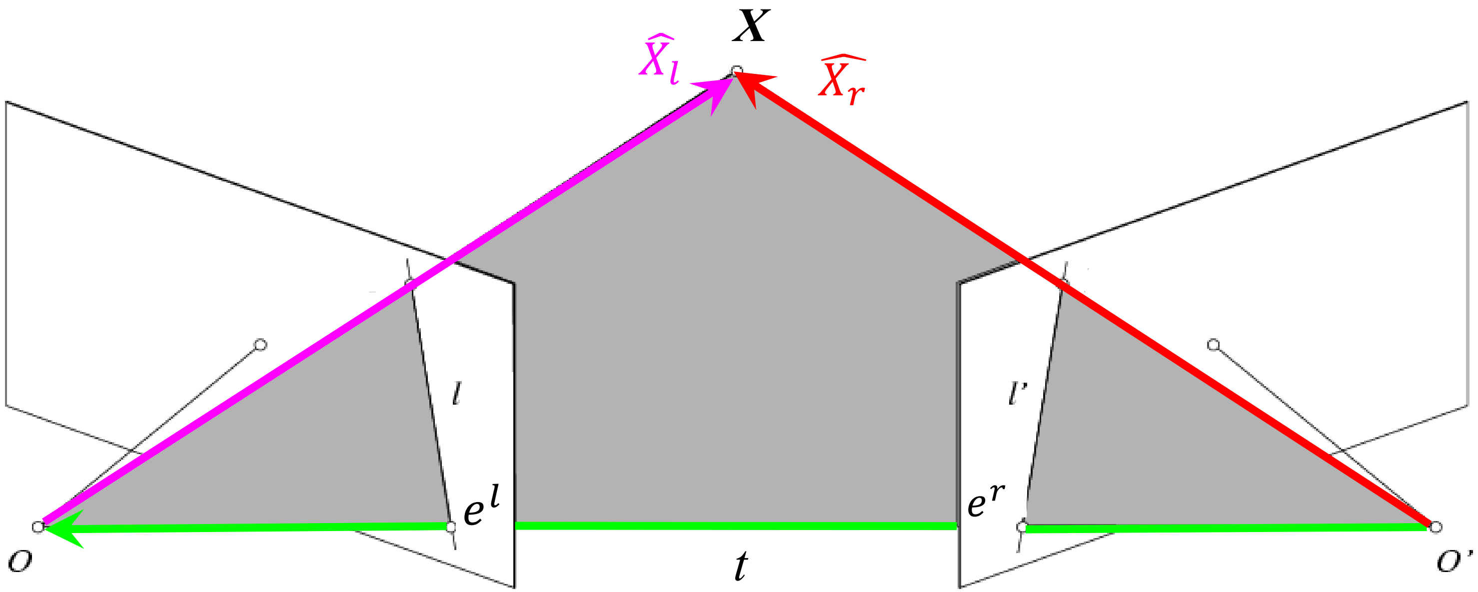

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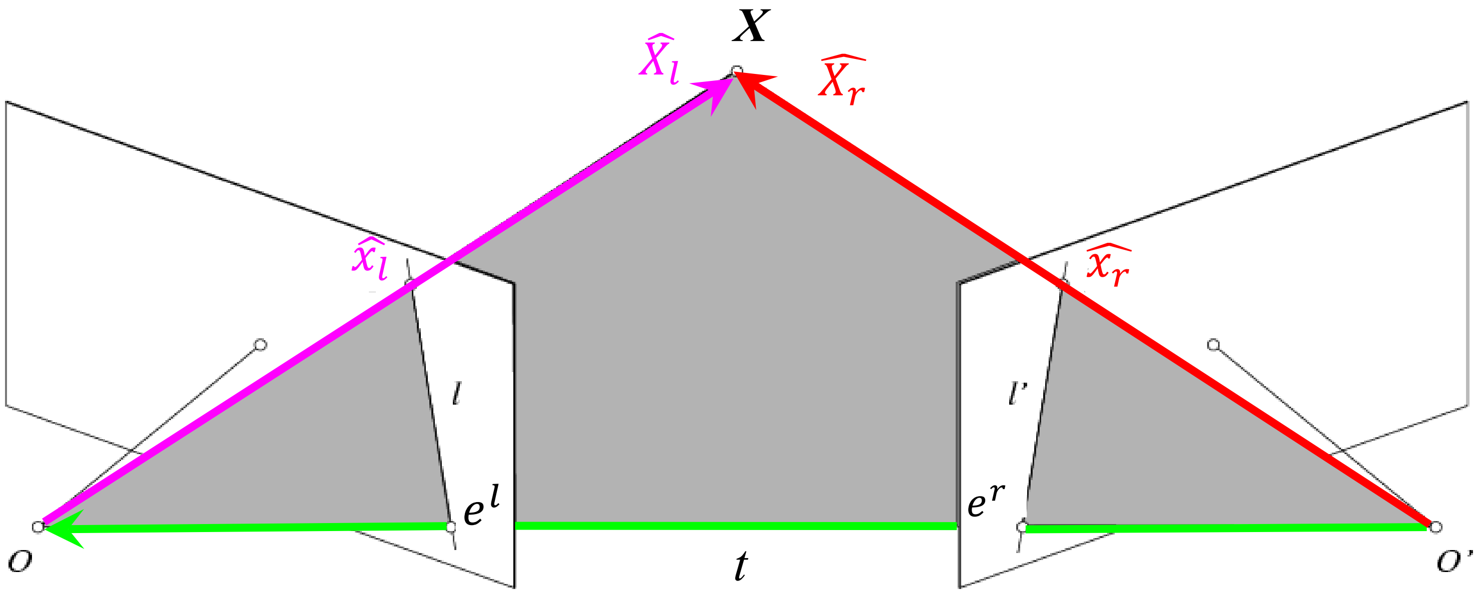

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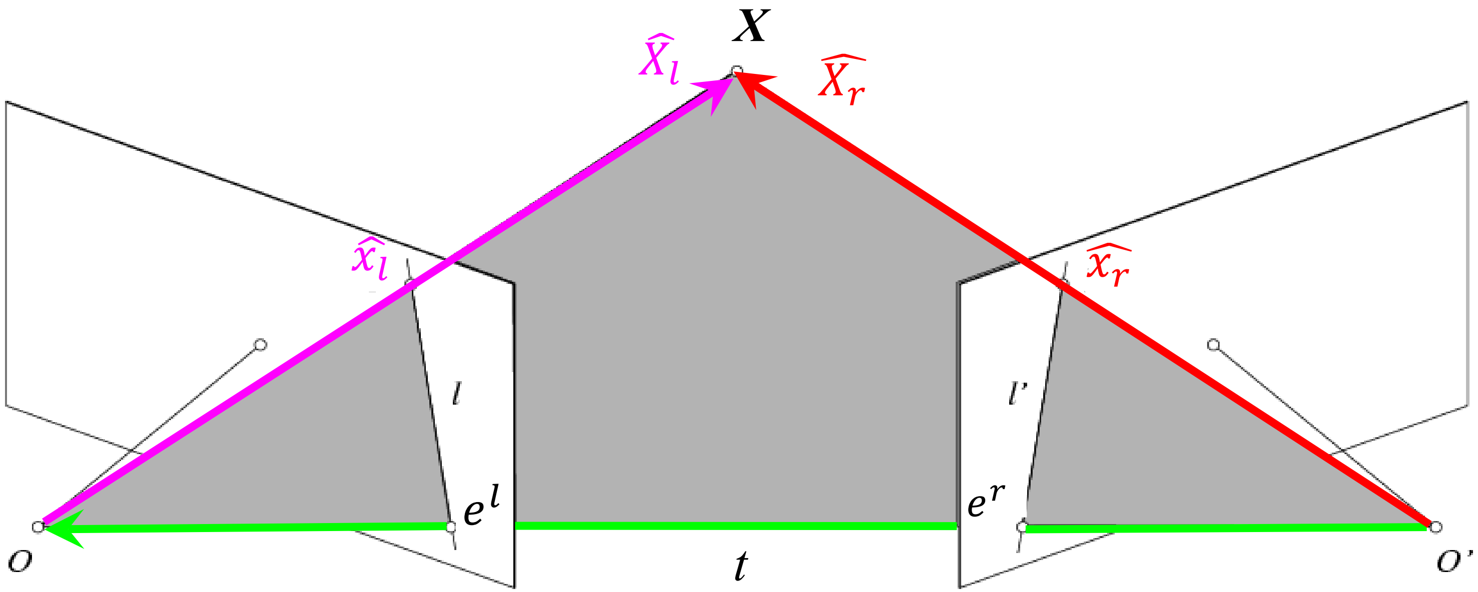

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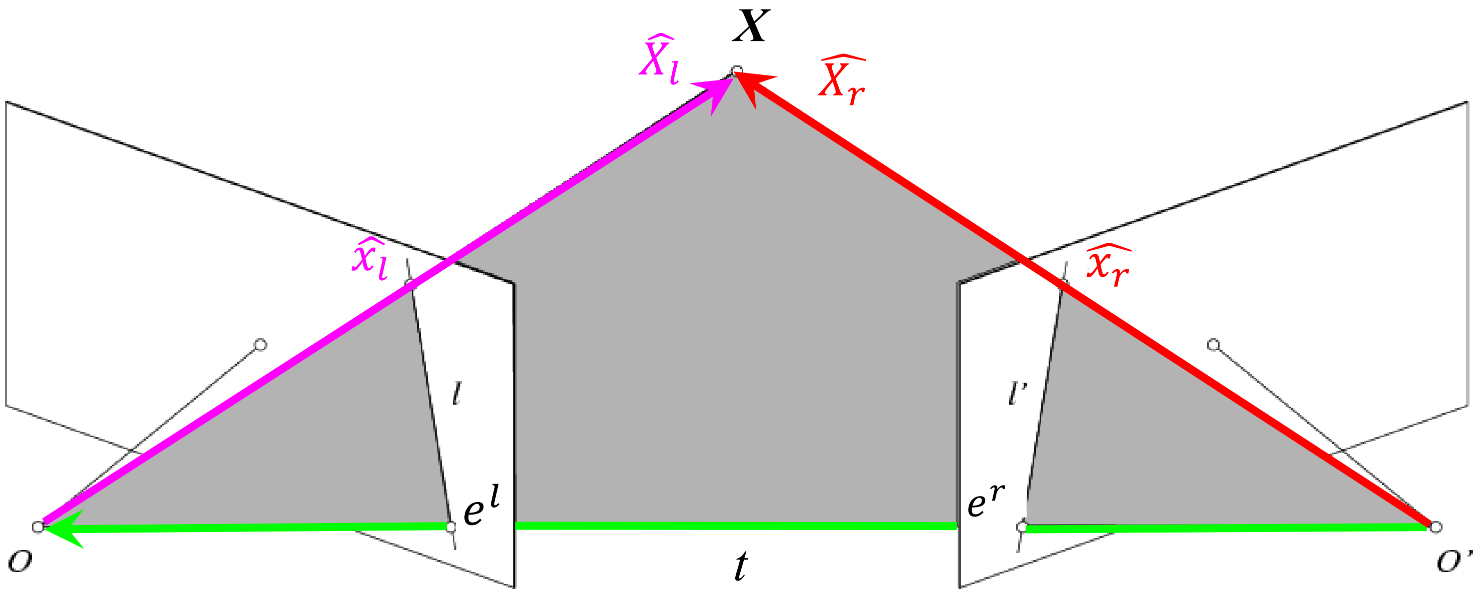

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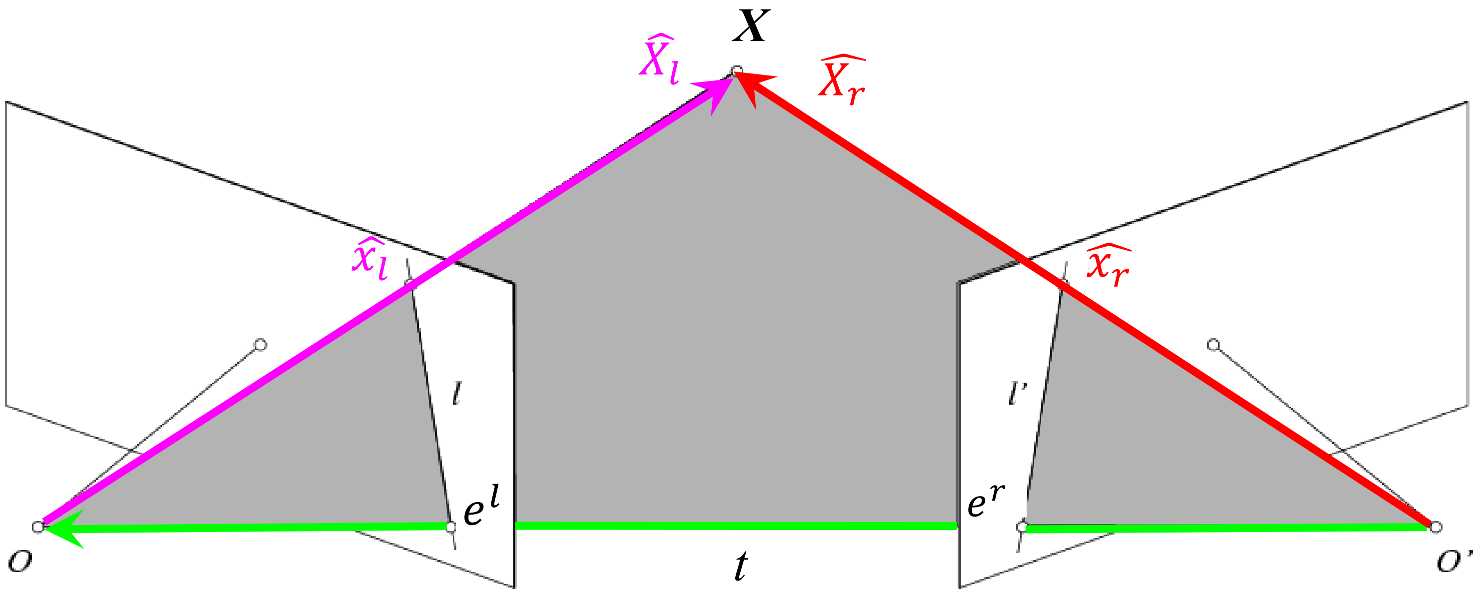

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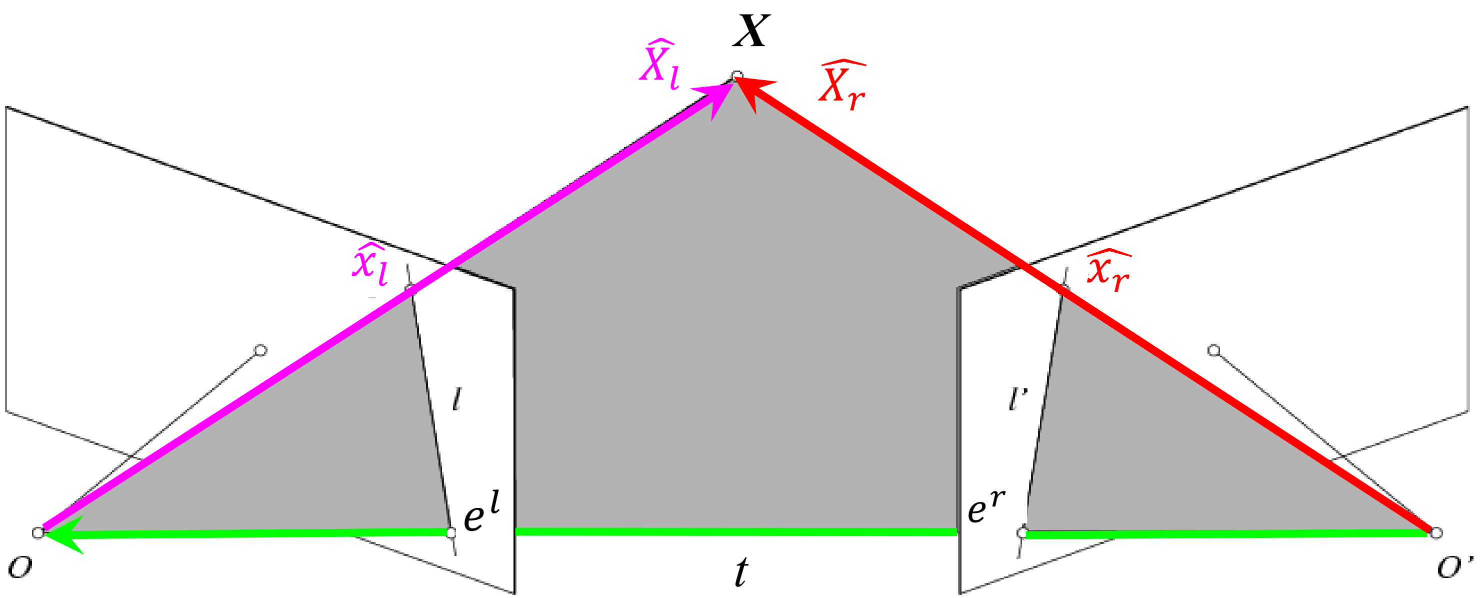

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Line representation in homogenous coordinate

- Line equation: ax + by + c = 0

$$line_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$$

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  - Can represent a line with vector [a b c]
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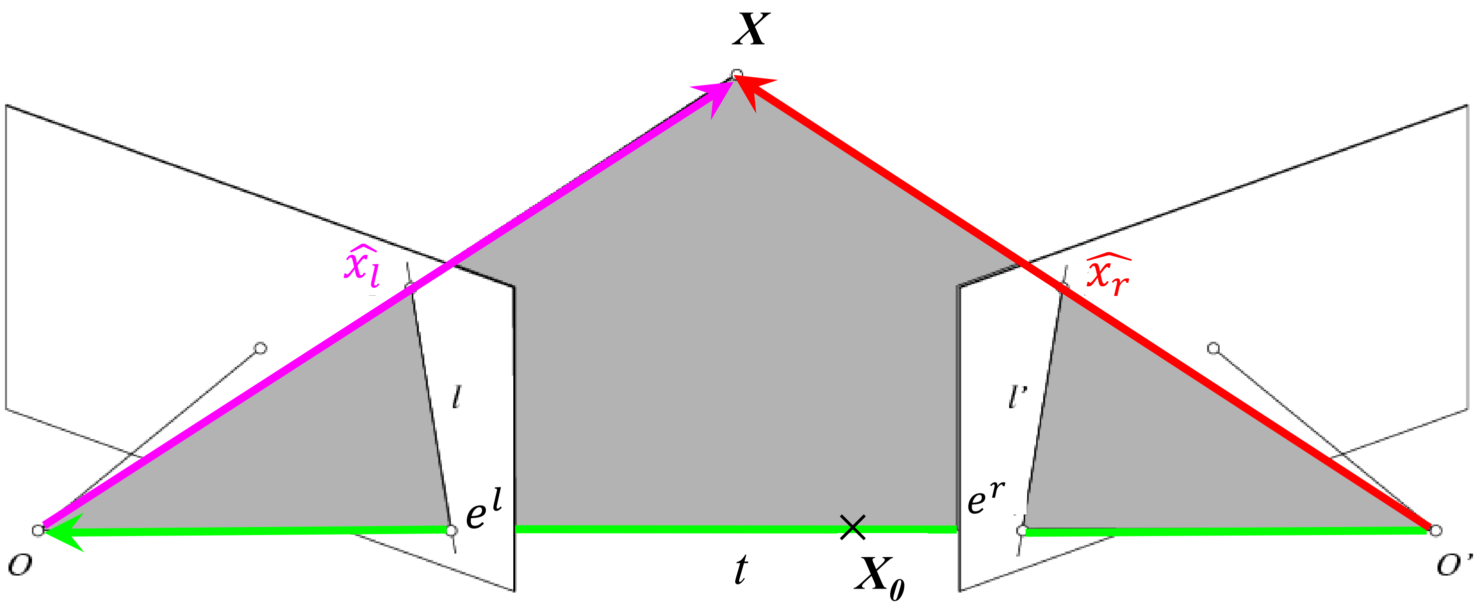
Epipoles and fundamental matrix



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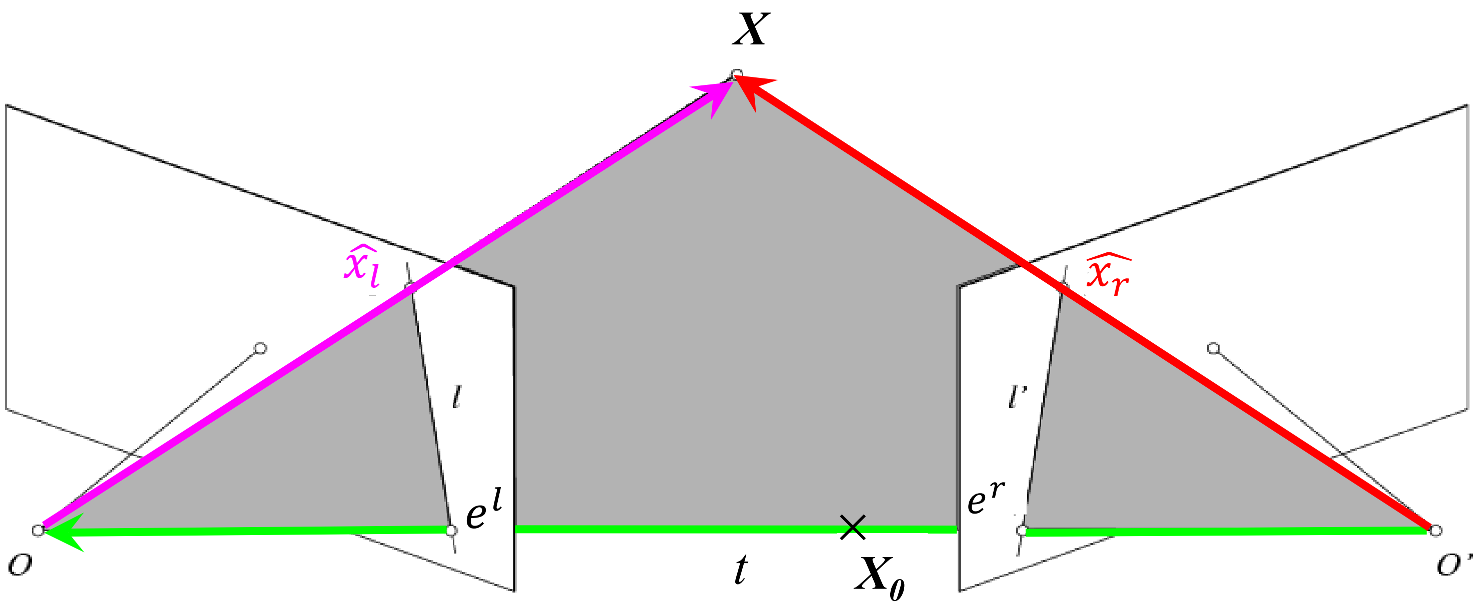

$\hat{x}_l^T \underbrace{F\hat{x}_r}_{\text{A line passing through } \hat{x}_l} = 0$ and recall that $Fe^r = F^T e^l = 0$

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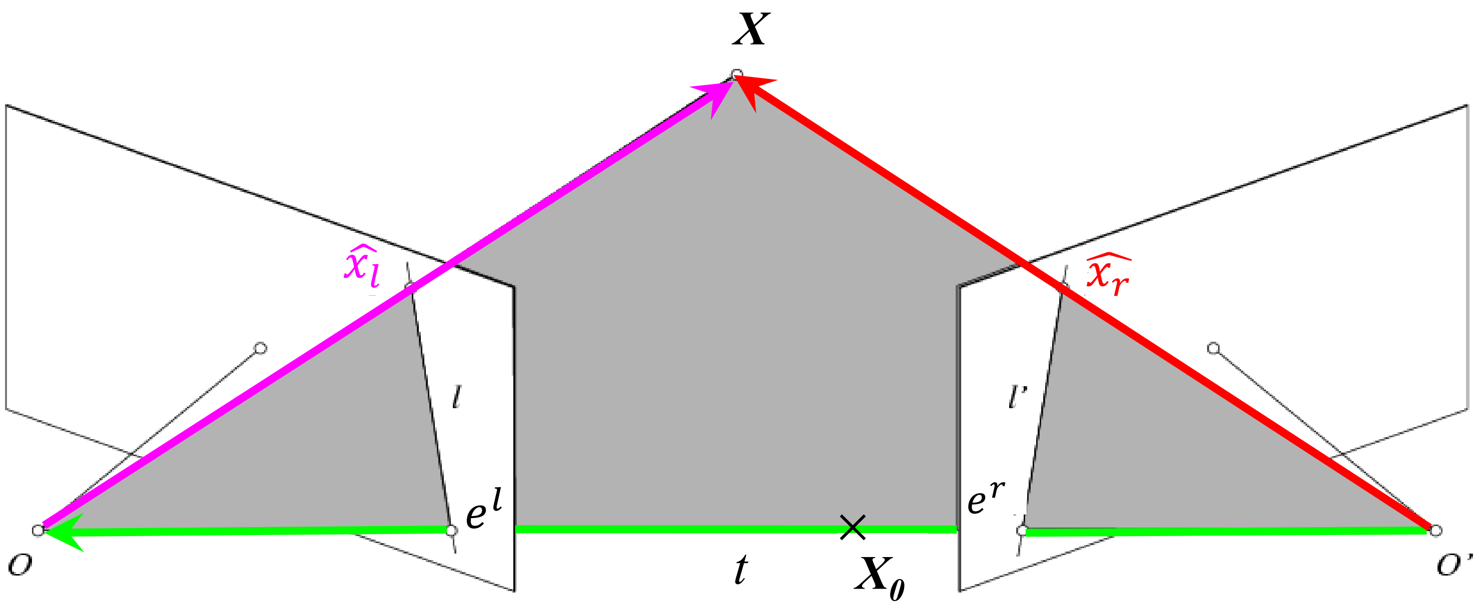

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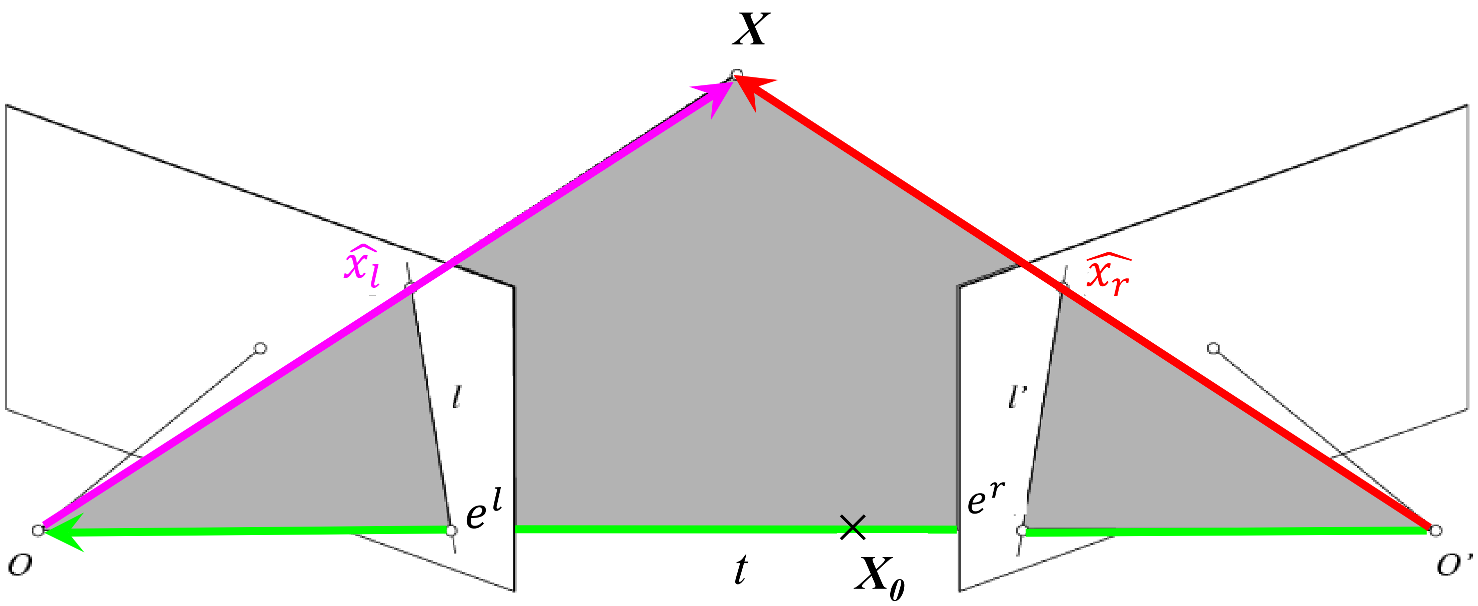

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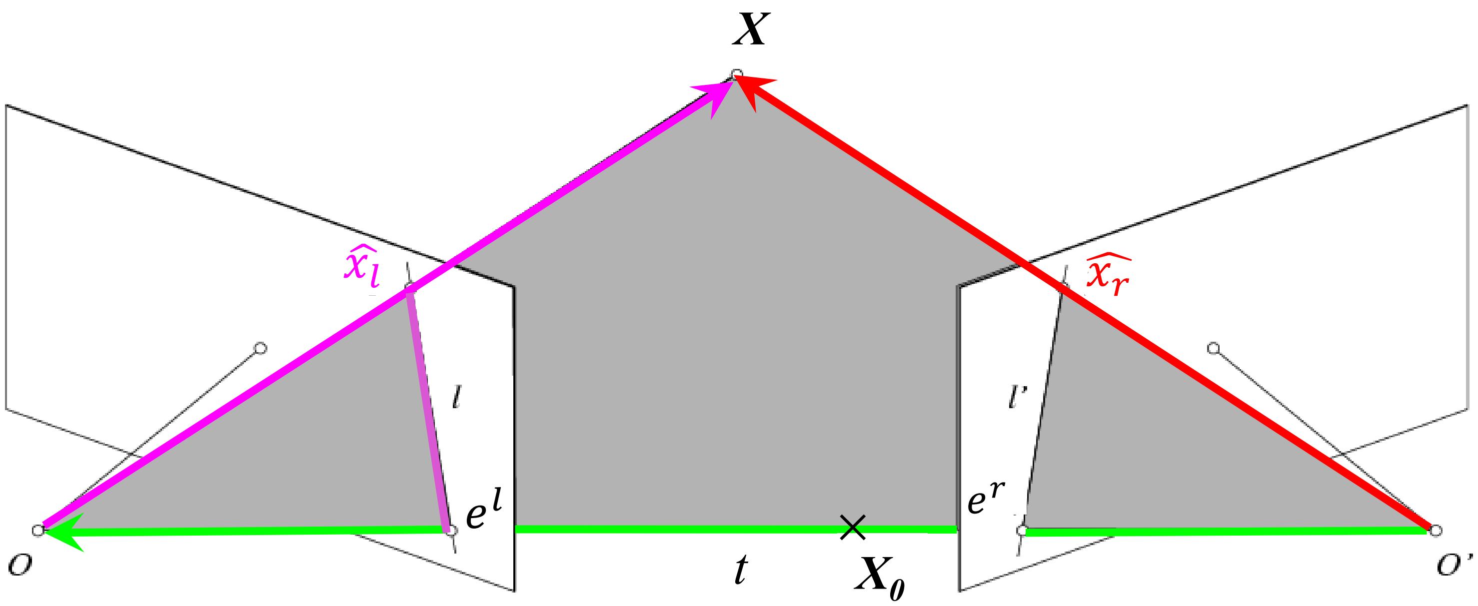
$F\hat{x}_r$ passes through not just $\hat{x}_l$ but also the epipole e

Epipoles and fundamental matrix



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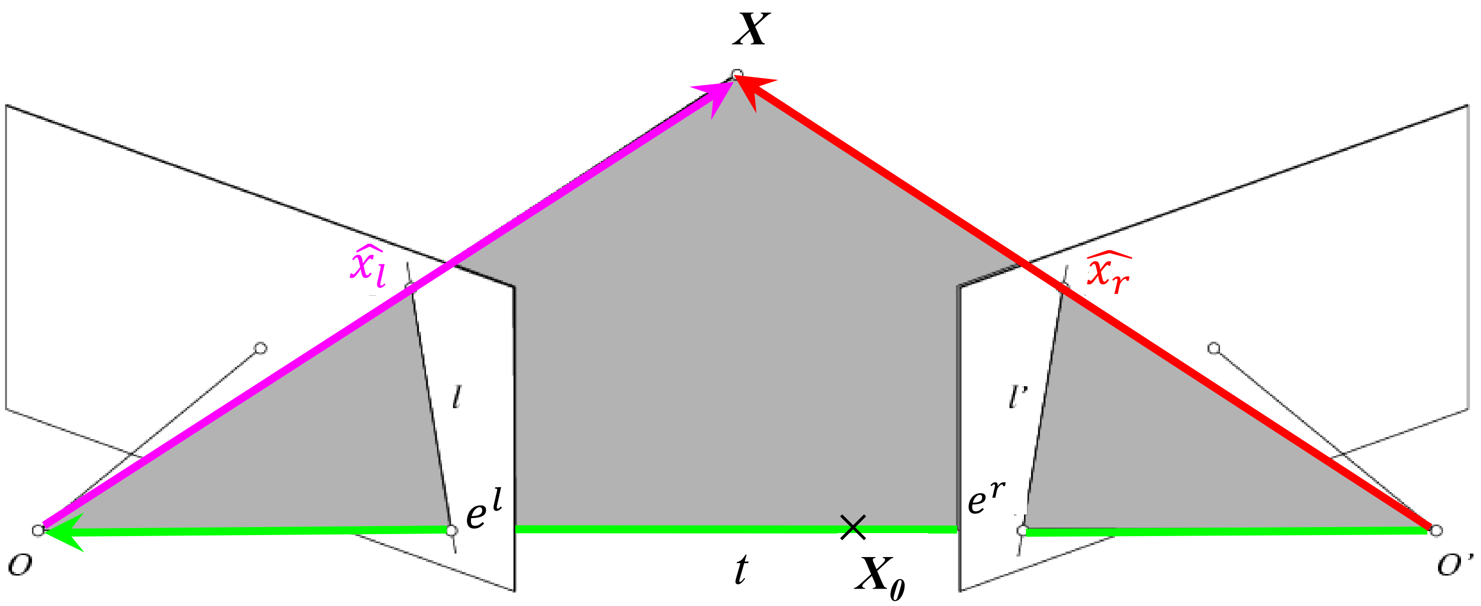
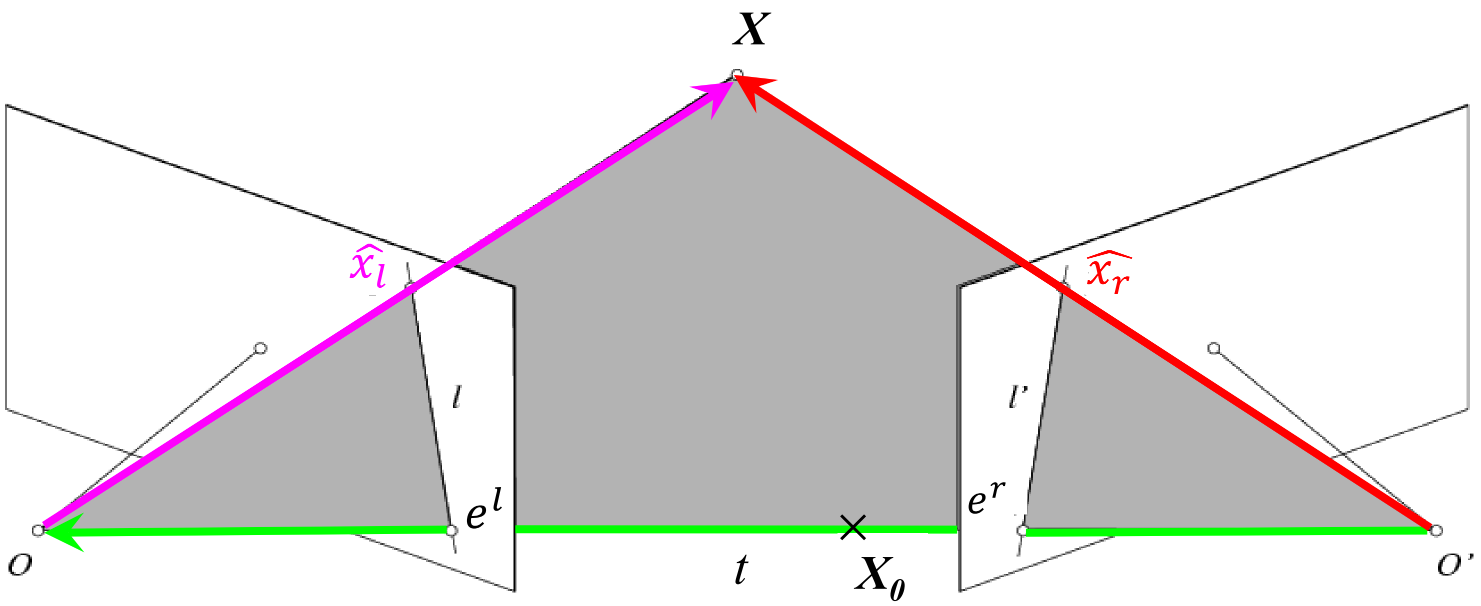
Thus, it is actually the epipolar line l

Epipoles and fundamental matrix



Similarly, $\underbrace{\hat{x}_r^T \quad F^T \hat{x}_l}_{\text{A line passing through } \hat{x}_r} = 0$ and recall that $F e^r = F^T e^l = 0$

Epipoles and fundamental matrix



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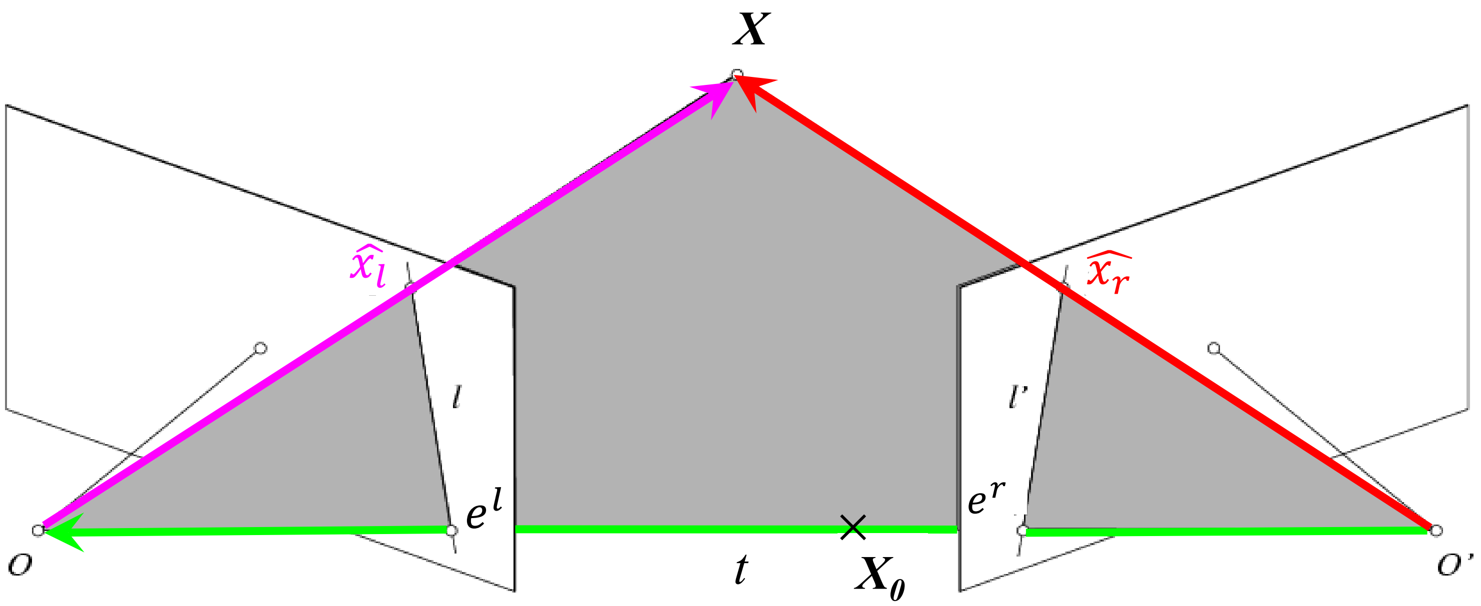
Epipoles and fundamental matrix

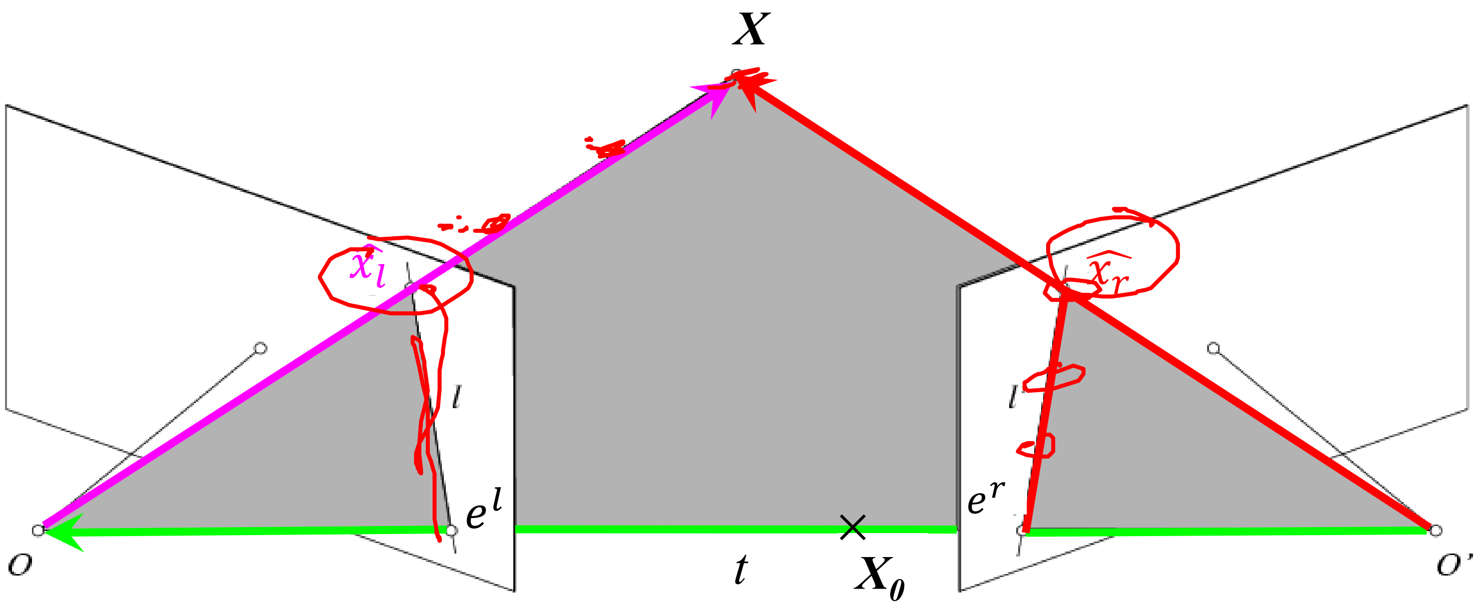


Similarly, $\hat{x}_r^{\,T} \underbrace{F^T \hat{x}_l}_{\text{A line passing through } \hat{x}_r} = 0$ and recall that $Fe' = F^T e^l = 0$

$$\Rightarrow e^r F^T \hat{x}_l = (F e^r)^T \hat{x}_l = 0$$

$F^T \hat{x}_l$ passes through not just $\hat{x}_r$ but also the epipole e'

Epipoles and fundamental matrix



Similarly, $\underbrace{\hat{x}_r^T \quad F^T \hat{x}_l}_{}$ $= 0$ and recall that $Fe' = F^T e^l = 0$

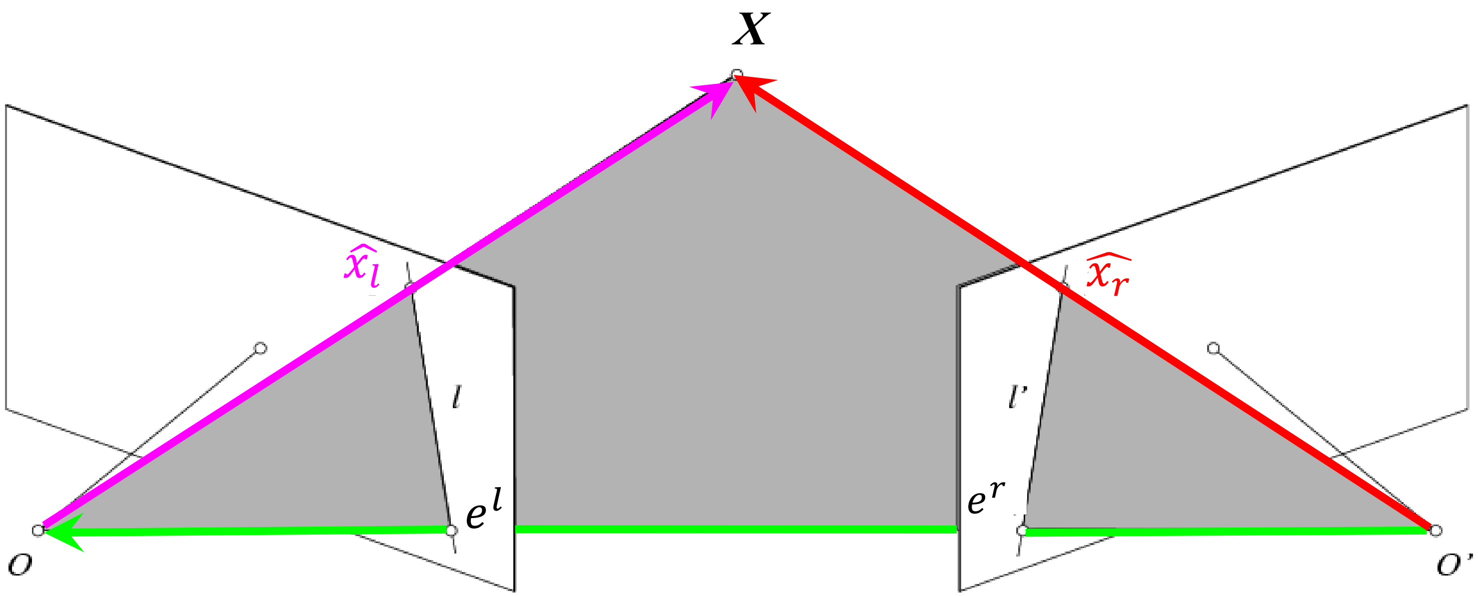
$F^T \hat{x}_l$: A line passing through $\hat{x}_r$

$\Rightarrow e^r F^T \hat{x}_l = (F e^r)^T \hat{x}_l = 0$

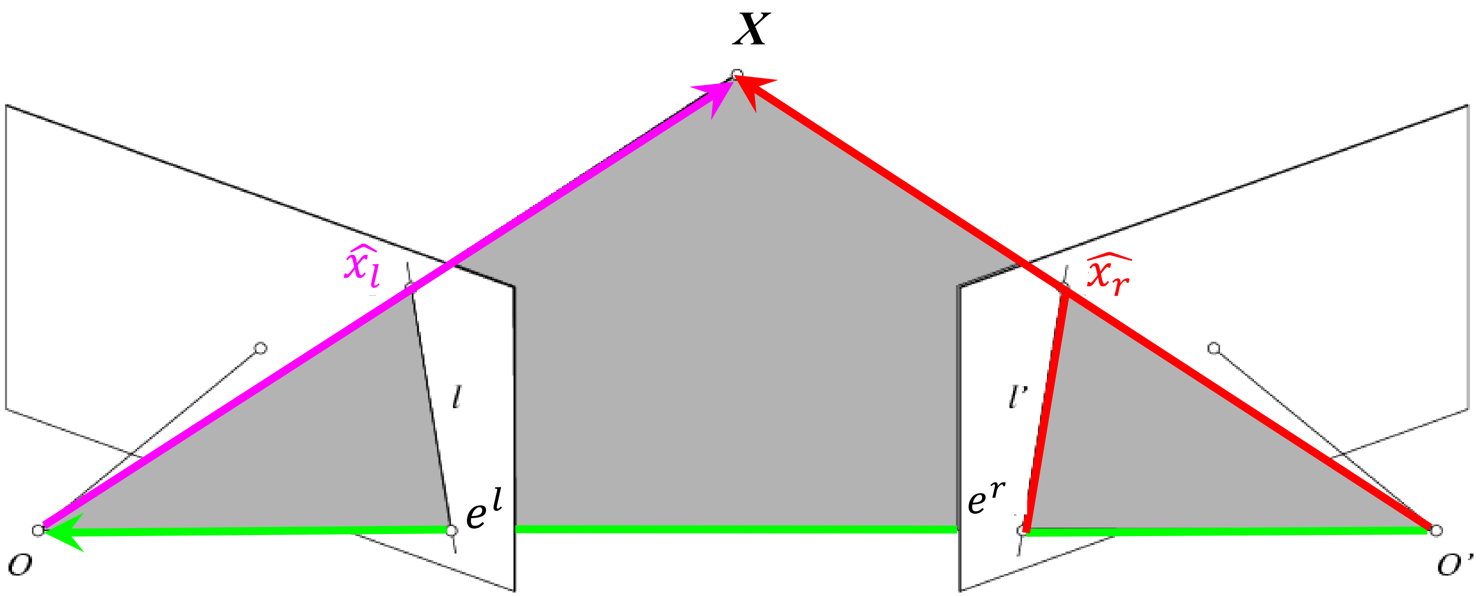
$F^T \hat{x}_l$ passes through not just $\hat{x}_r$ but also the epipole e'

Thus, it is actually the epipolar line l'

Epipoles and fundamental matrix

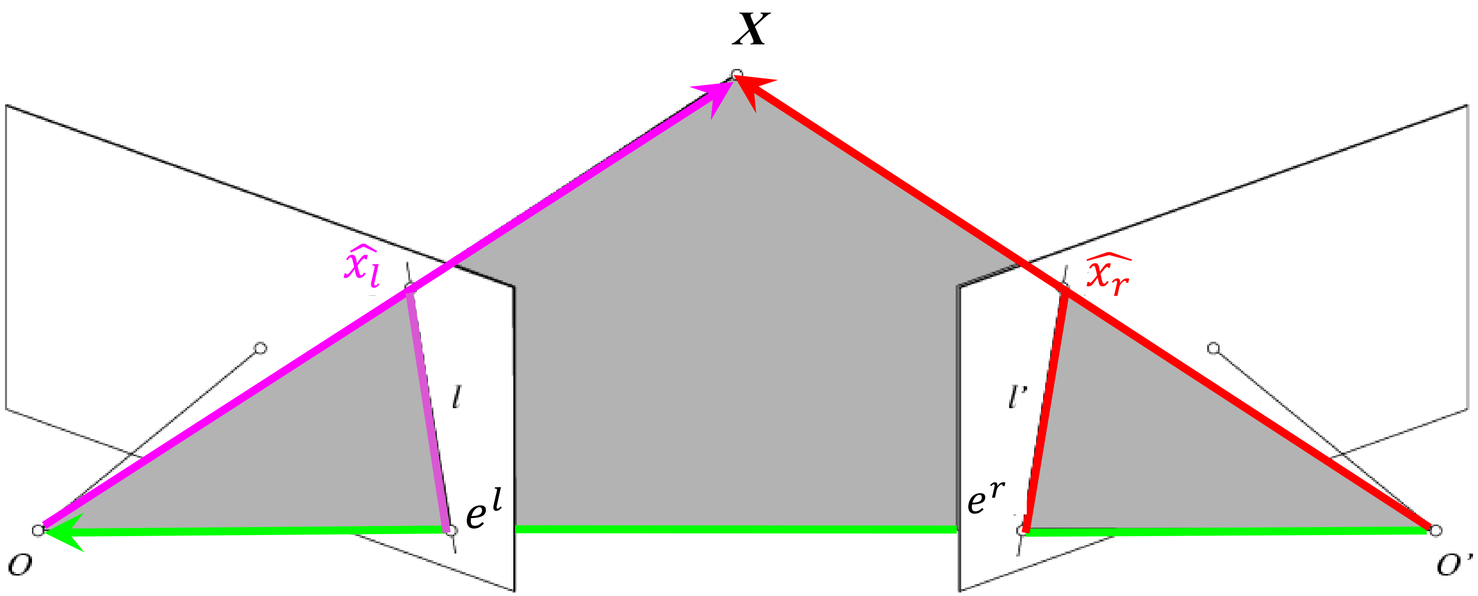


Epipoles and fundamental matrix



$F\hat{x}_r$ is the epipolar line l

Epipoles and fundamental matrix



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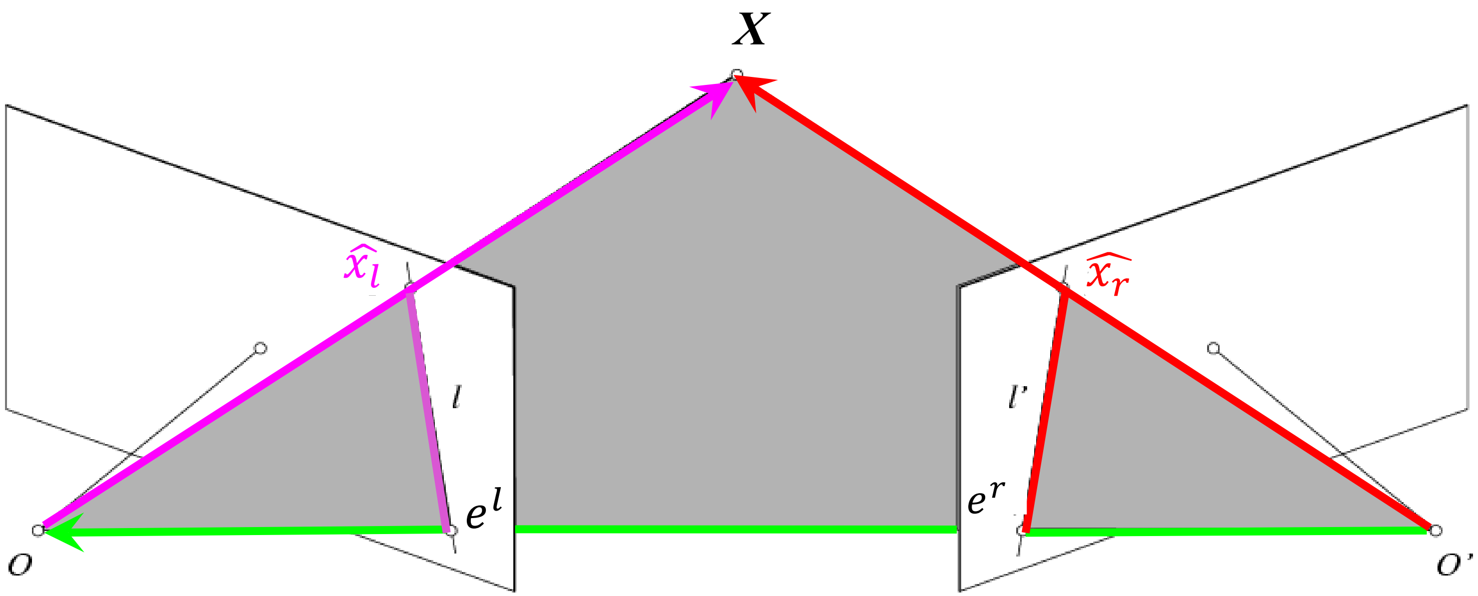
Epipoles and fundamental matrix



$F\hat{x}_r$ is the epipolar line l

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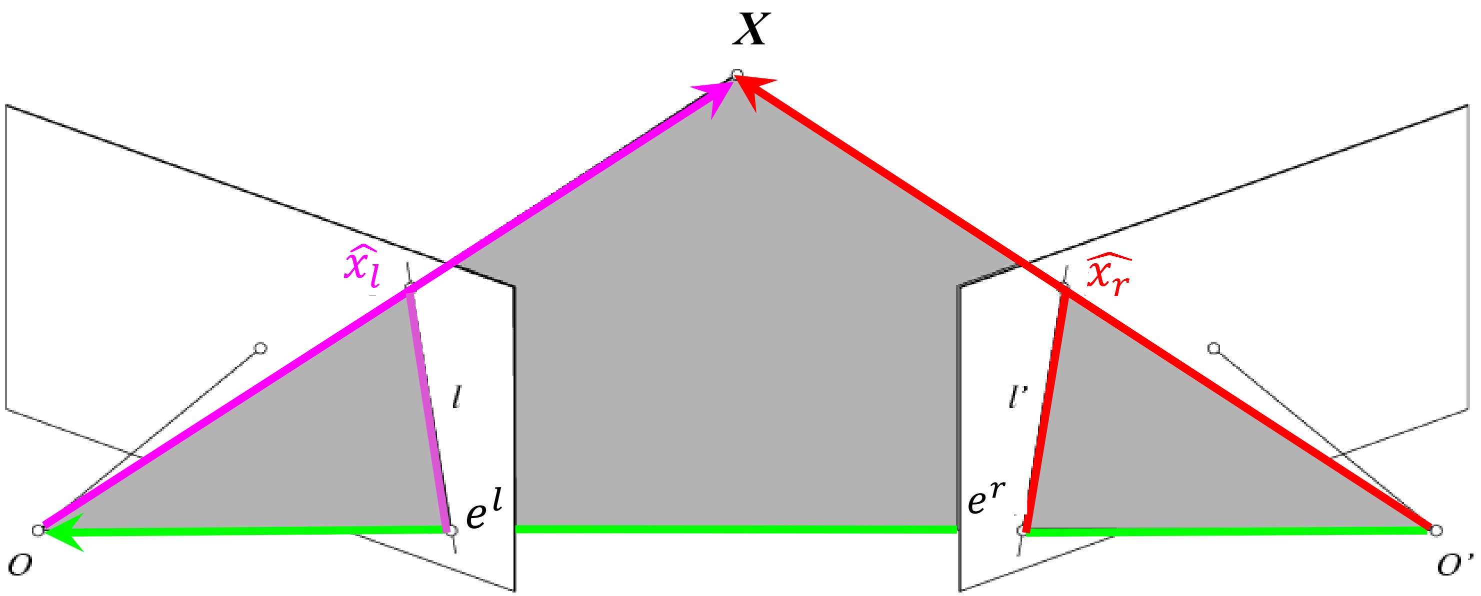
Given $\hat{x}_r$ on the right view, we just need to search $F\hat{x}_r$ on the left

Epipoles and fundamental matrix



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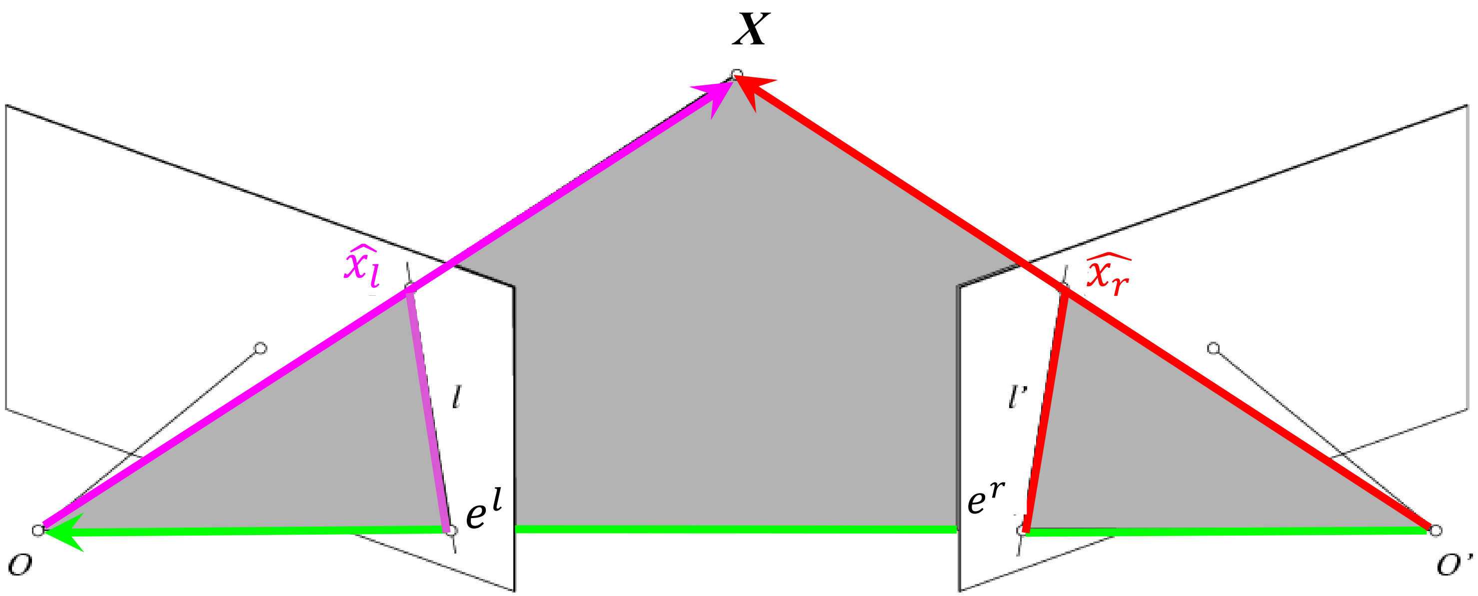
Epipoles and fundamental matrix



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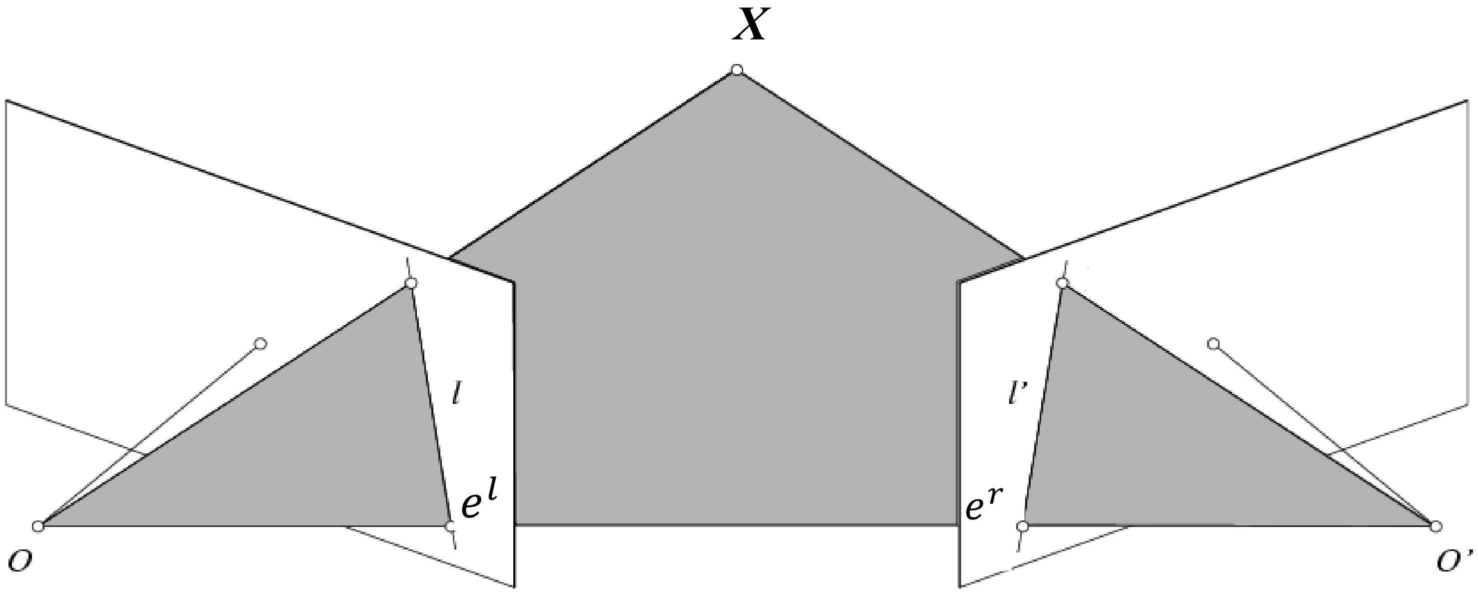
Given $\hat{x}_l$ on the right view, we just need to search $F^T\hat{x}_l$ on the left

This restrict our search space from 2D to 1D

Fundamental matrix



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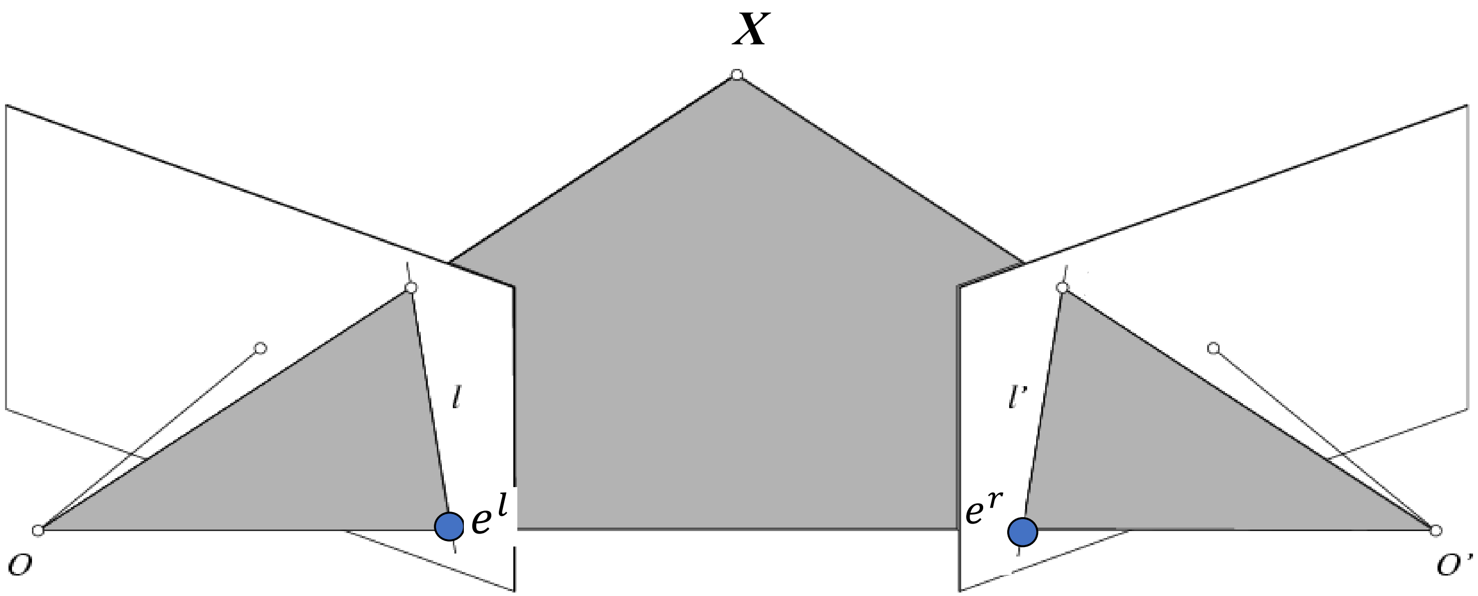
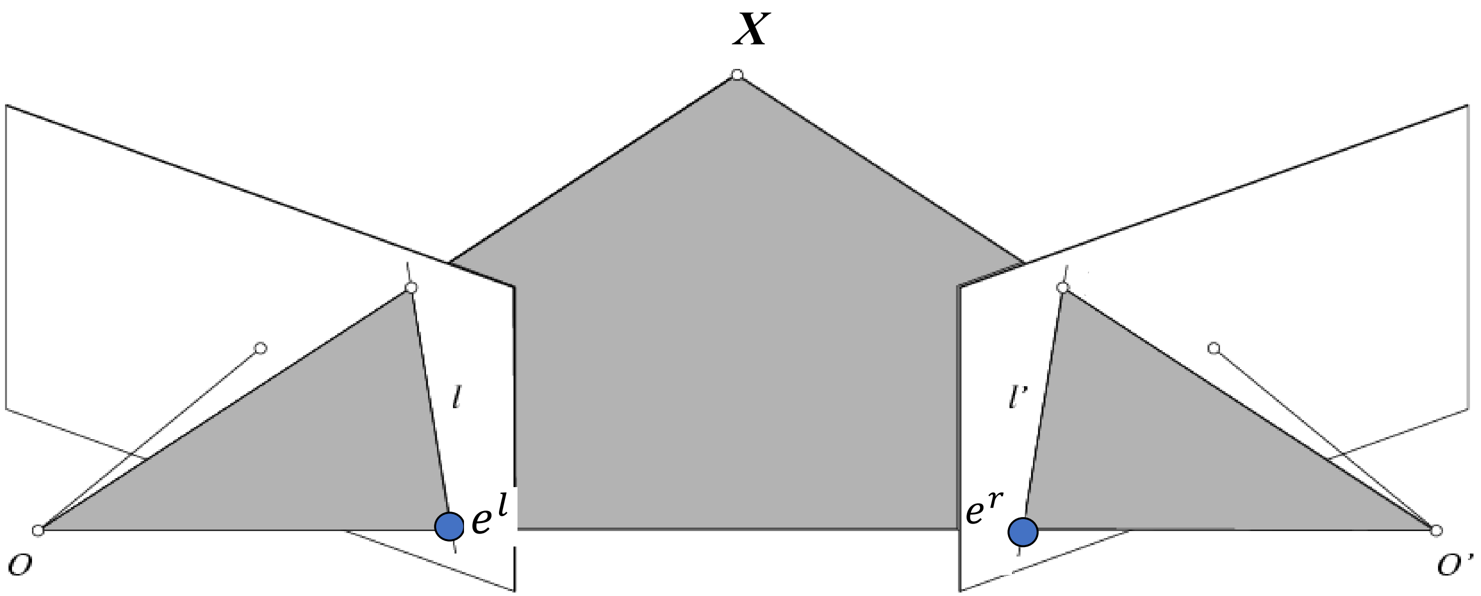


- $F e^r = 0$ and $F^T e^l = 0$ (nullspaces of $F = e^r$; nullspace of $F^T = e^l$)

Fundamental matrix



- $F e^r = 0$ and $F^T e^l = 0$ (nullspaces of $F = e^r$; nullspace of $F^T = e^l$)
- F is singular (rank two): det(F)=0
- F has seven degrees of freedom: 9 entries but defined up to scale, det(F)=0

Remark

- Since $F = K_r^{-T} E K_l'^{-1}$

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When $K_r = K'_l = I,\ F = E$

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When $K_r = K'_l = I$, $F = E$, and $\hat{x}_l^T E \hat{x}_r = 0$

Remark

- Since $$F = K_r^{-T} E K_l'^{-1}$$

  When $K_r = K'_l = I$, $F = E$, and $\hat{x}_l^T E \hat{x}_r = 0$

- This is sometimes how essential matrix is introduced

Estimating the Fundamental Matrix

- 8-point algorithm
  - Least squares solution using SVD on equations from 8 pairs of correspondences
  - Enforce det(F)=0 constraint using SVD on F

Estimating the Fundamental Matrix

- 8-point algorithm
  - Least squares solution using SVD on equations from 8 pairs of correspondences
  - Enforce det(F)=0 constraint using SVD on F

Note: estimation of F (or E) is degenerate for a planar scene.

8-point algorithm

1. Solve a system of homogeneous linear equations
   a. Write down the system of equations

$$\mathbf{x}^T F \mathbf{x}' = 0$$

$$uu'f_{11} + uv'f_{12} + uf_{13} + vu'f_{21} + vv'f_{22} + vf_{23} + u'f_{31} + v'f_{32} + f_{33} = 0$$

$$\mathrm{A}\boldsymbol{f} = \begin{bmatrix} u_1u_1{}' & u_1v_1{}' & u_1 & v_1u_1{}' & v_1v_1{}' & v_1 & u_1{}' & v_1{}' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_v' & u_nv_n{}' & u_n & v_nu_n{}' & v_nv_n{}' & v_n & u_n{}' & v_n{}' & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ \vdots \\ f_{33} \end{bmatrix} = \mathbf{0}$$

8-point algorithm

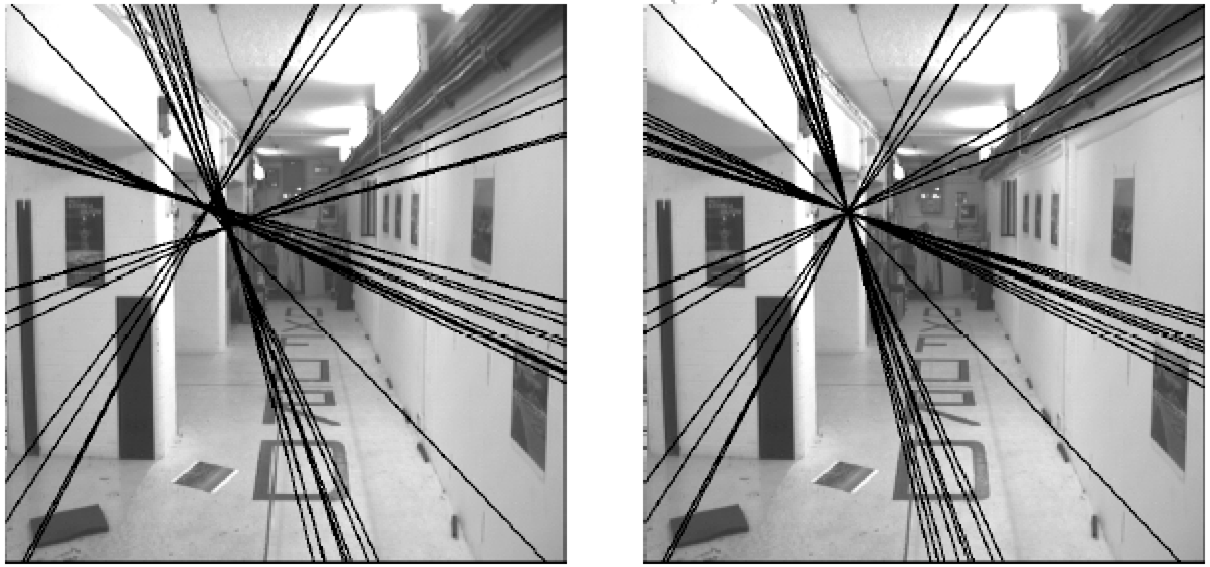
1. Solve a system of homogeneous linear equations
   a. Write down the system of equations
   b. Solve **f** from A**f**=**0** using SVD

Matlab:

```
[U, S, V] = svd(A);
f = V(:, end);
F = reshape(f, [3 3])';
```

Need to enforce singularity constraint

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



Left : Uncorrected $\mathtt{F}$ – epipolar lines are not coincident.

Right : Epipolar lines from corrected $\mathtt{F}$.

8-point algorithm

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   Matlab:
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   [U, S, V] = svd(A);
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   ```

2. Resolve det(F) = 0 constraint using SVD

   Matlab:
   ```
   [U, S, V] = svd(F);
   S(3,3) = 0;
   F = U*S*V';
   ```

Problem with eight-point algorithm

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Problem with eight-point algorithm

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48

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- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters.

From E to get back R and t

- If we decompose E using svd to $USV^\top$
- Let's define $W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- One can verify that $[t]_\times = VWSV^\top$ and $R = UW^\top V^\top$ is a pair of valid estimate of R and t such that $R[t]_\times = UW^\top V^\top VWSV^\top = USV^\top = E$
  - How do we get t back from $[t]_\times$?
- Note that there are more solutions
  - E.g., Replace W by $W^\top$
  - Note that only one solution will satisfy chirality (reconstructed points in front of the camera)

Compute 3D point location

- Given R, t, and observed points in the two views. We can construct two lines that should contain the 3D point. And the point should be just the interception
- Denote two lines as $a_1 + \lambda_1 b_1$ and $a_2 + \lambda_2 b_2$. We can solve the interception by solving
- $\min_{\lambda,\mu} \|a_1 + \lambda_1 b_1 - a_2 - \lambda_2 b_2\|^2$

$$\Rightarrow \begin{cases} \lambda_1 \|b_1\|^2 + (a_1 - a_2)^\top b_1 - \lambda_2 b_1^\top b_2 = 0 \\ \lambda_2 \|b_2\|^2 - (a_1 - a_2)^\top b_2 - \lambda_1 b_1^\top b_2 = 0 \end{cases}$$

- The final point will be

$$\frac{a_1 + \lambda_1 b_1 + a_2 + \lambda_2 b_2}{2}$$

Caveat

- Note that many textbooks use the convention of $E = [t]_\times R$ rather than $E = R[t]_\times$ as derived here
  - Both are correct just they first rotate and then translate the axis rather than first translate and then rotate the axis as shown earlier
  - But consequently, the t there is different as below
- Note that for any y, $[R\ t]_\times y = R\left([t]_\times R^\top y\right)$ since crossing $R\ t$ with y is the same as crossing t with $R^\top y$ and then rotate by R
  - $\Rightarrow [R\ t]_\times = R\ [t]_\times R^\top$
  - $\Rightarrow E = R[t]_\times = (R[t]_\times R^\top)R = [Rt]_\times R$
  - Therefore, for the alternative formulation $E = [t]_\times R$, we have the translation there actually equal to Rt instead in our notations

Summary

- $X_l^T E X_r$=0
- $E = R[t]_\times$
- $Fe^r = F^T e^l = 0$
- $\hat{x}_l{}^T F \hat{x}_r = 0$
- $F^T \hat{x}_l = l'$
- $F \hat{x}_r = l$
- Estimating the fundamental matrix is a kind of “weak calibration”