Meta Learning

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Meta Learning: Learn to learn

- Learn meta-knowledge that shares among tasks
- Often associate with few-shot learning

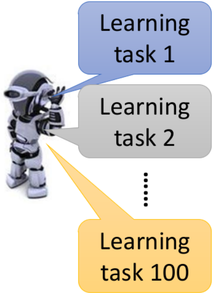

Image credit: Hung-yi Lee

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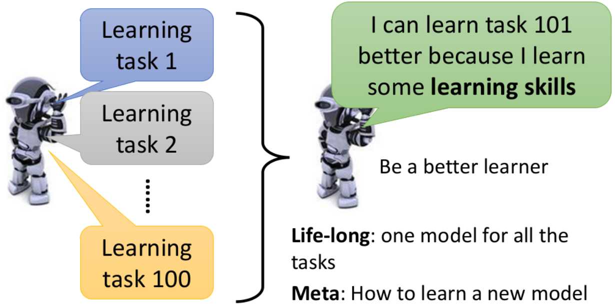

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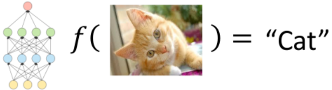
Meta Learning vs Machine Learning

- Machine learning: given input and output, find a function f that maps input to output



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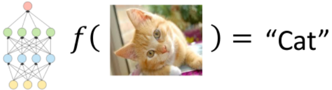
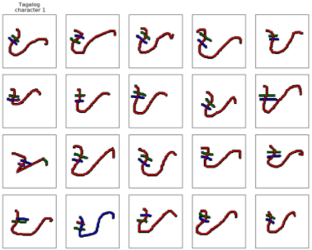


- Meta learning: given task training data, find a function F that maps training data to a good ML function $\boldsymbol{f}$

Image credit: Hung-Yi Lee

MNIST of Meta Learning: Omniglot

- 1623 characters
- 20 examples each
- Github



Other datasets: miniImageNet, CUB

Jargons

- N-ways K-shots: N classes and K samples each

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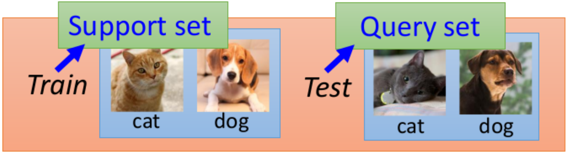
- In each task,



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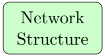
- Focus only on F with same network structure but different initialization ϕ



Gradient-based Approach

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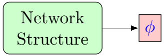

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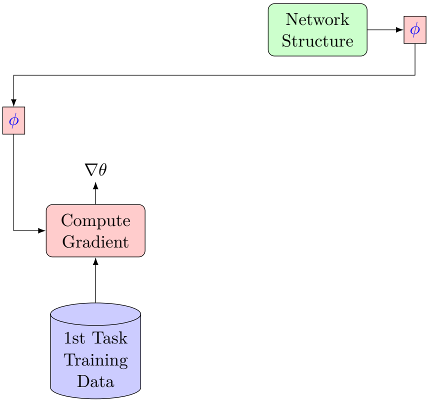

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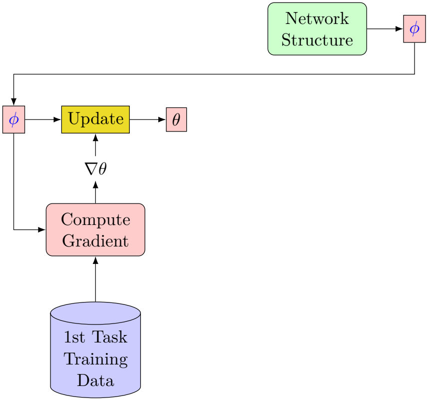

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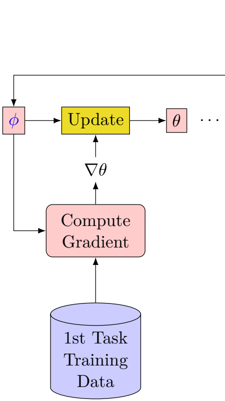

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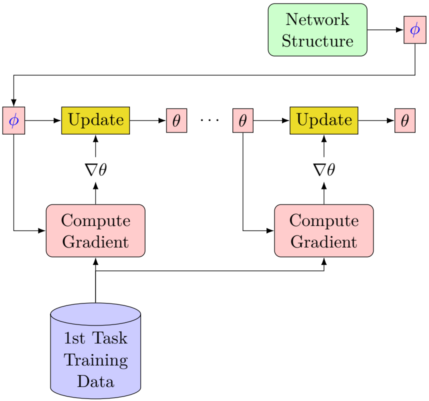

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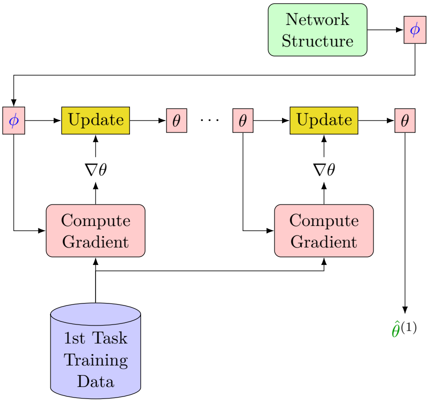

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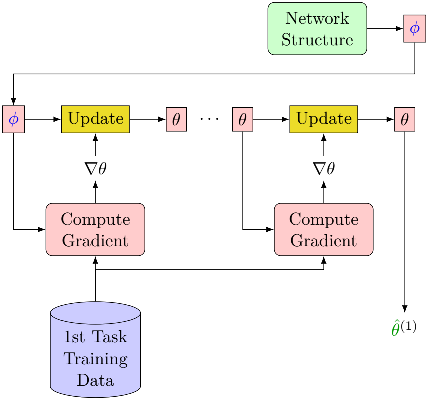

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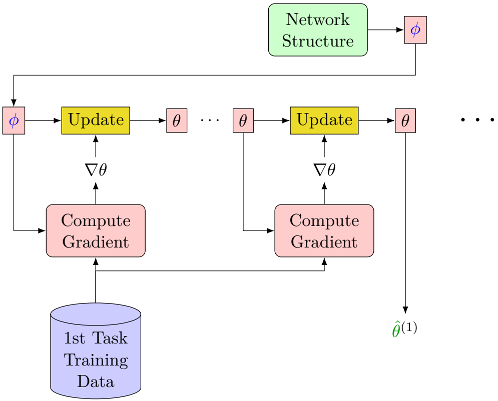

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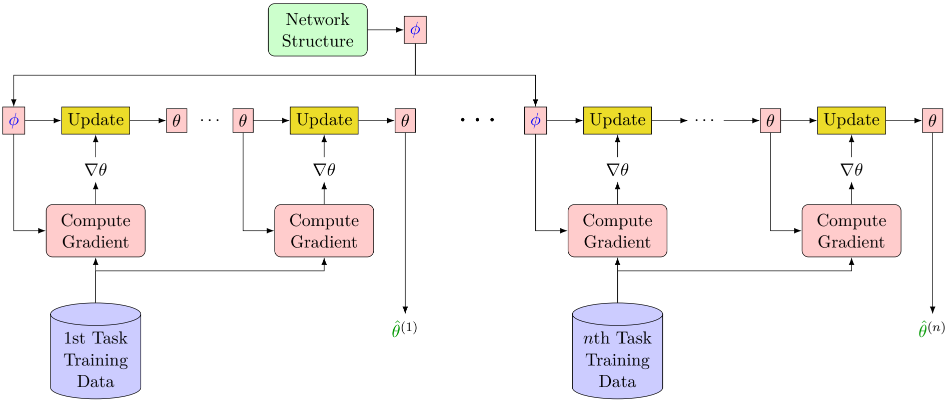

Gradient-based Approach

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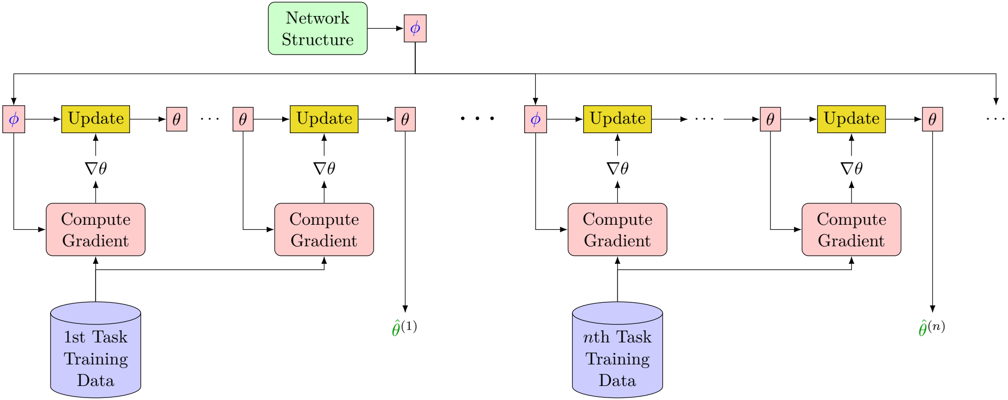
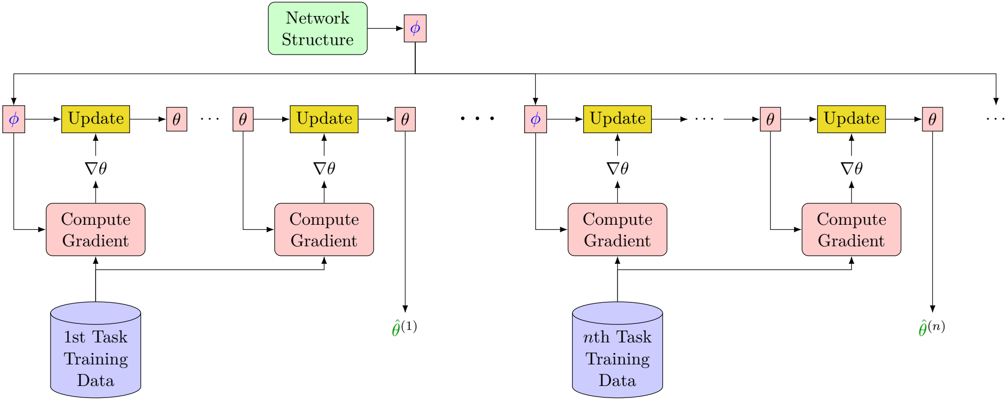


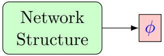
Gradient-based Approach

- Focus only on F with same network structure but different initialization ϕ
- Minimize $L(\phi) = \sum_n l^{(n)}(\hat{\theta}^{(n)})$ (c.f. $L(\phi) = \sum_n l^{(n)}(\phi)$ for pre-training)



(FO)MAML

- MAML (Model Agnostic Meta Learning): optimize task θ with only 1 gradient update
- FOMAML (First Order MAML): 1st order approximation. Get rid of 2nd order terms



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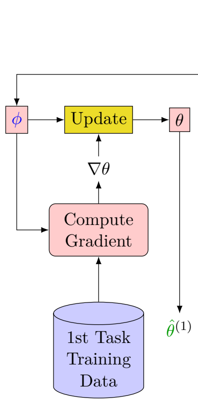

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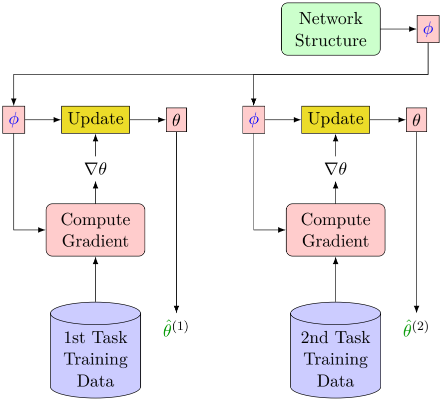

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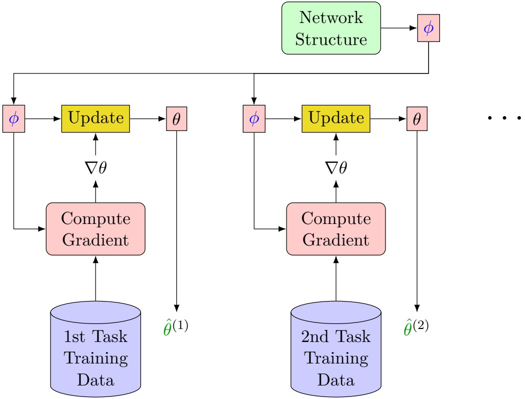

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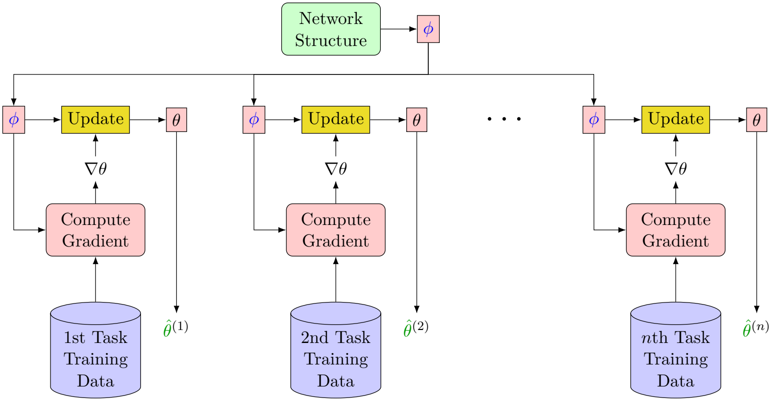

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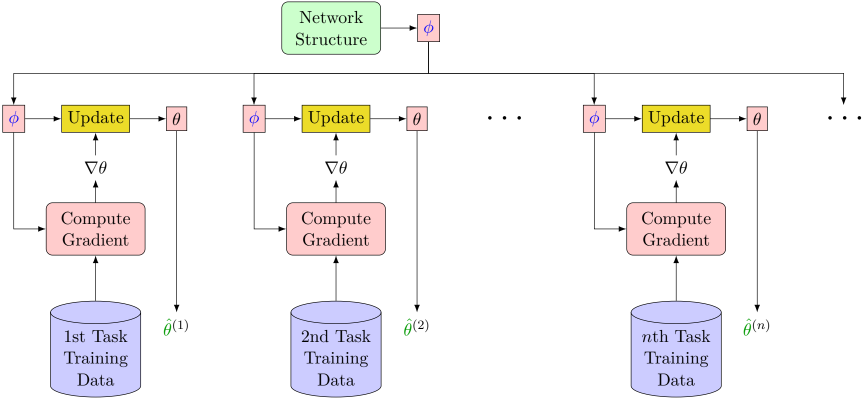

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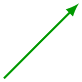
(FO)MAML vs Pretraining

MAML

Model Pre-training

ϕ_0 

 : sample from Task m

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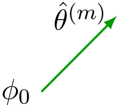

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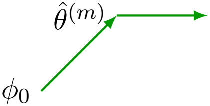

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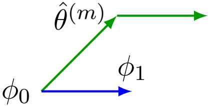

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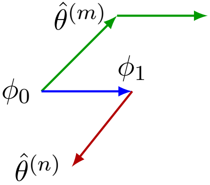

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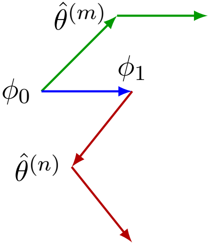

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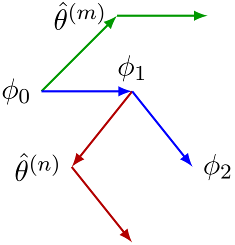

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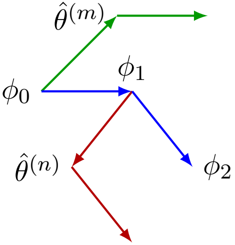
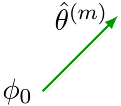


Model Pre-training

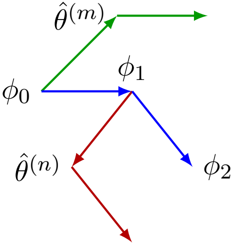


→ : sample from Task m

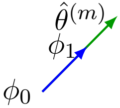
→ : sample from Task n

(FO)MAML vs Pretraining

MAML



Model Pre-training



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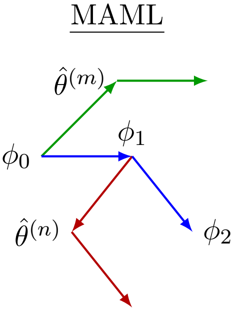

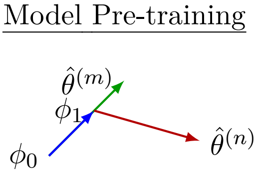

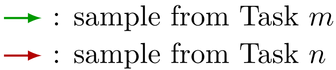

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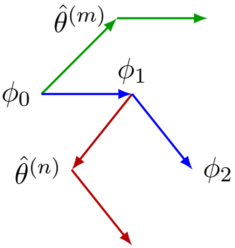

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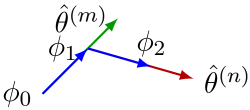

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MAML vs Pre-training

MAML optimizes the potential of ϕ: $L(\phi) = \sum_n l^{(n)}(\hat{\theta}^{(n)})$

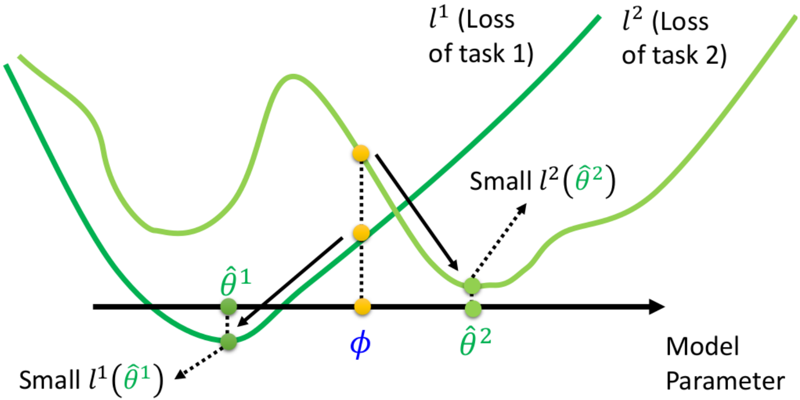


Image credit: Hung-yi Lee

MAML vs Pre-training

Pre-training optimizes the current ϕ for all tasks: $L(\phi) = \sum_n l^{(n)}(\phi)$

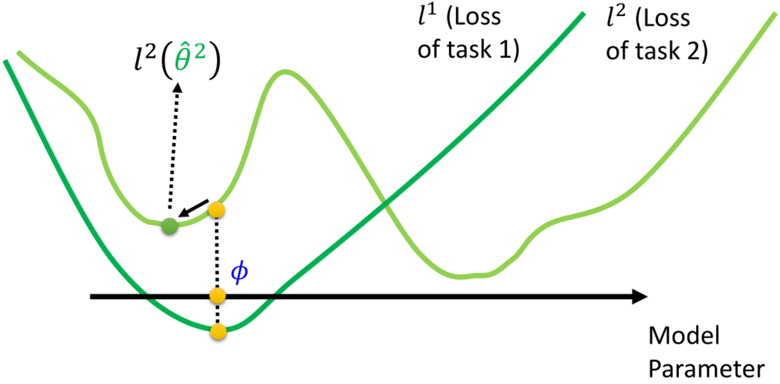


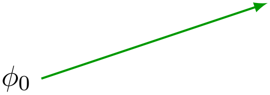
Image credit: Hung-yi Lee

Reptile

ϕ_0

→ : sample from Task m

Reptile



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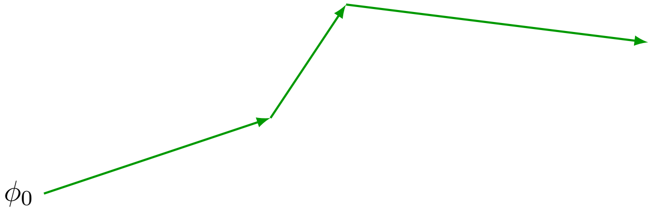
Reptile

Reptile



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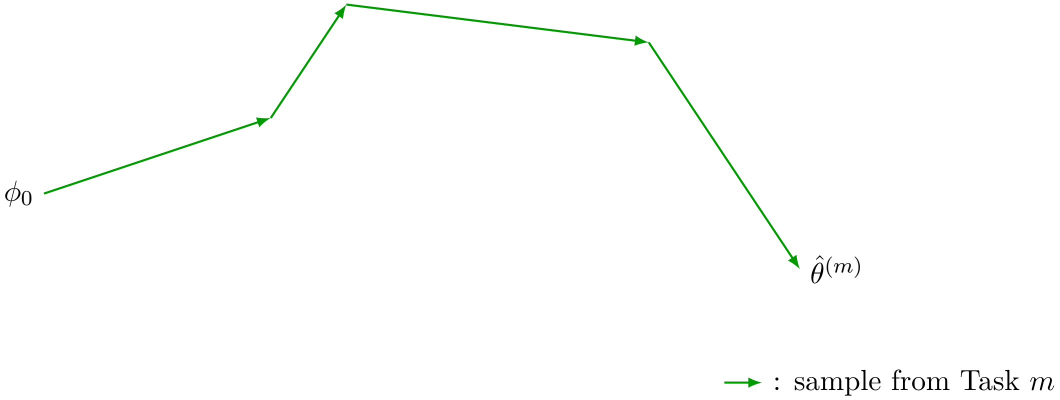
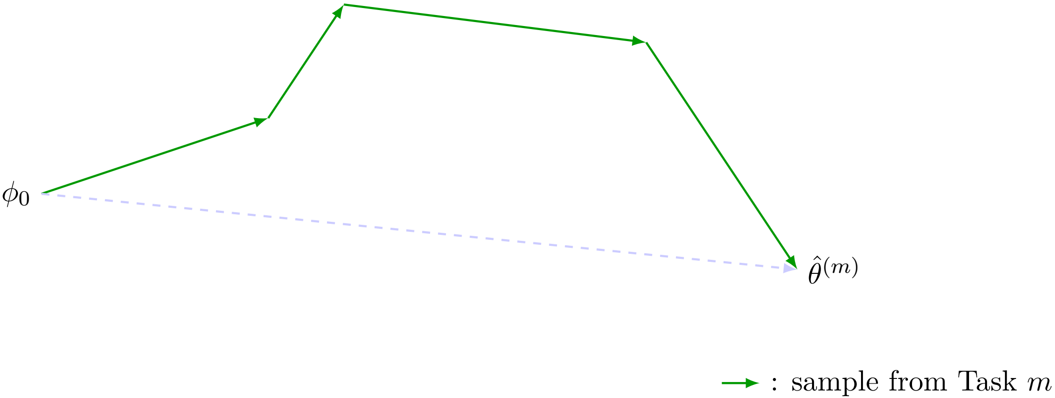
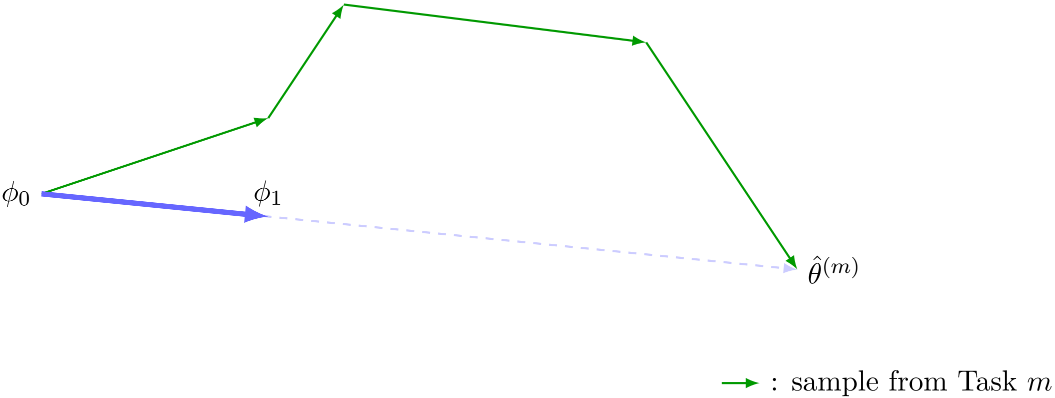
Reptile



Reptile




ϕ_0
ϕ_1
$\hat{\theta}^{(m)}$
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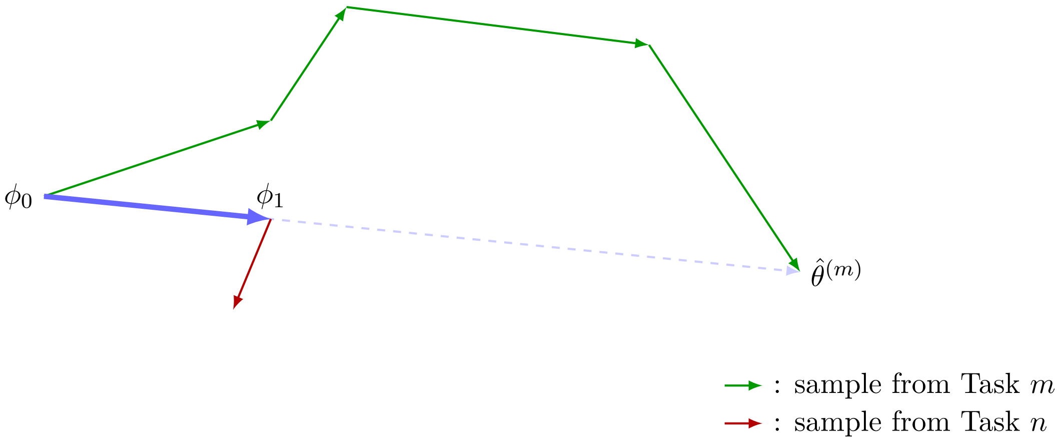
Reptile



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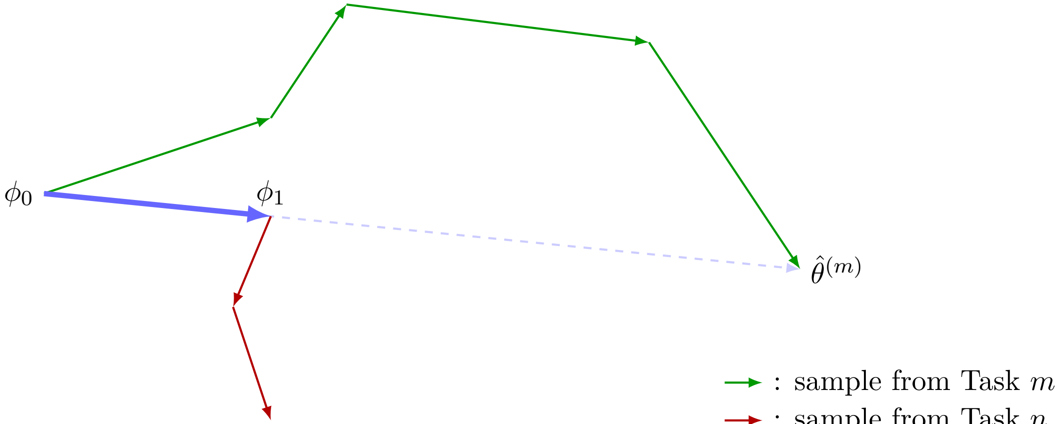

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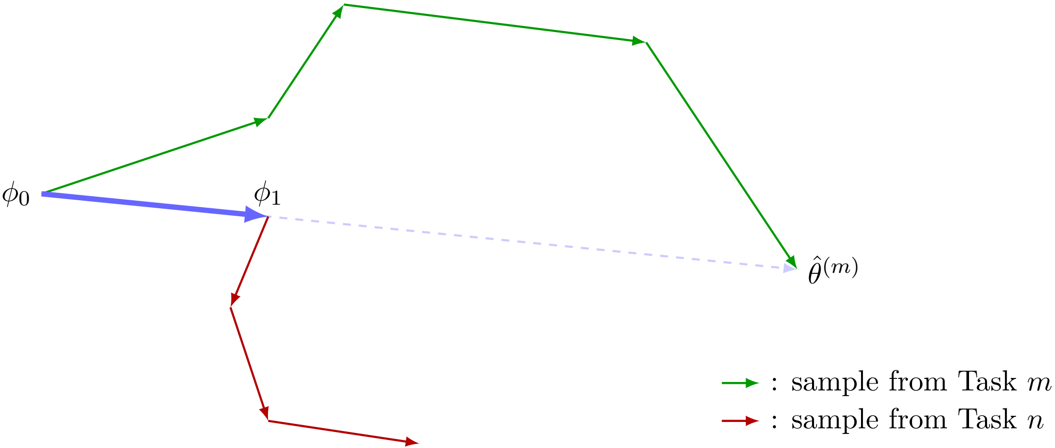

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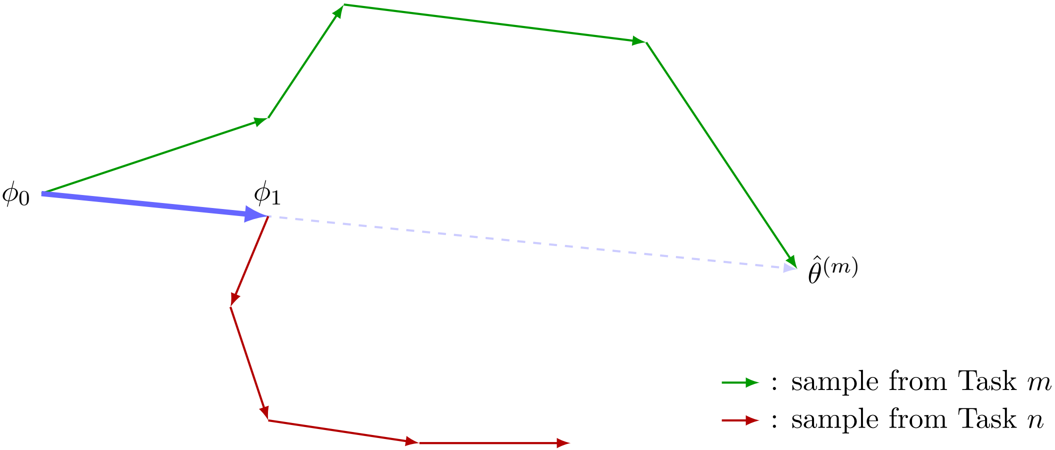

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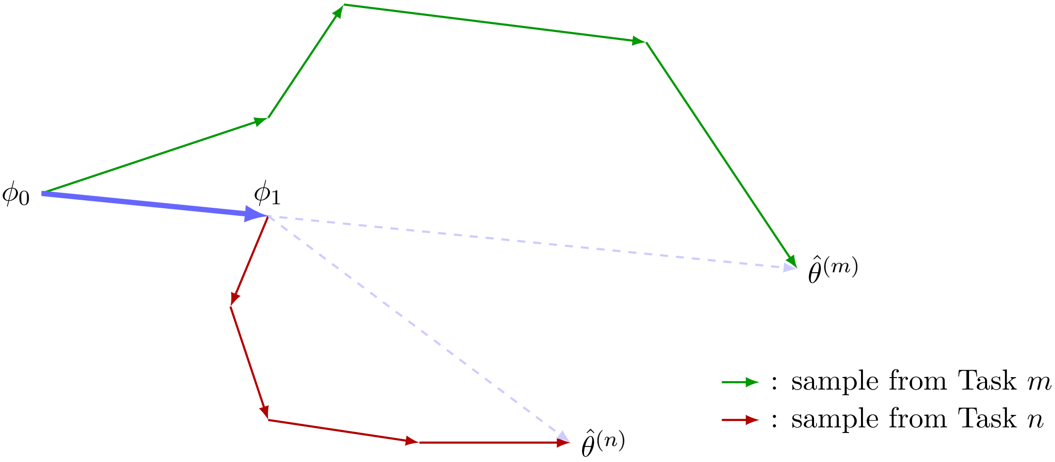

ϕ_0
ϕ_1
$\hat{\theta}^{(m)}$
$\hat{\theta}^{(n)}$
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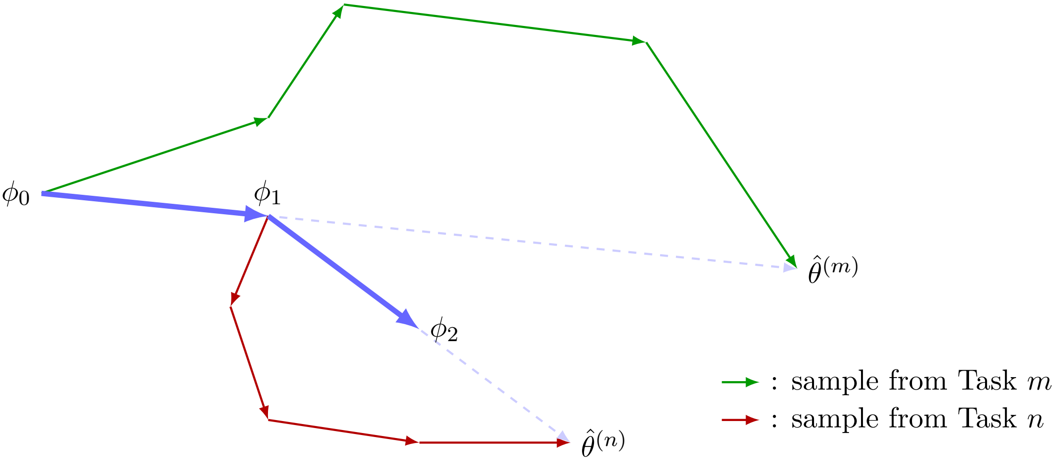
Reptile



ϕ_0
ϕ_1
ϕ_2
$\hat{\theta}^{(m)}$
$\hat{\theta}^{(n)}$
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Reptile Result

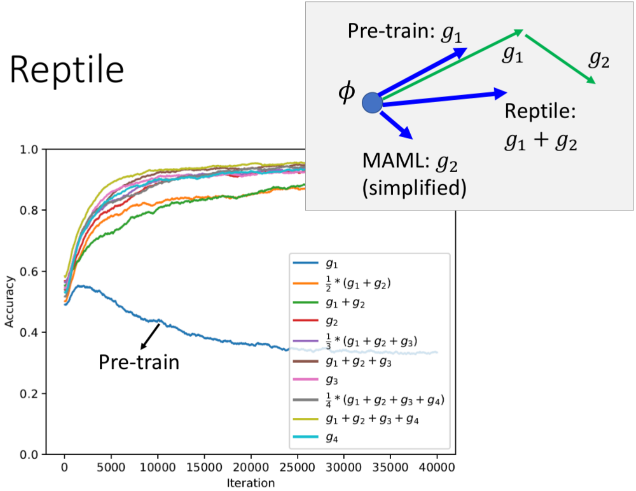

Image credit: Hung-yi Lee

iMAML: Implicit Gradient

For each task τ, we don't want the model parameter θ too far from the meta-parameter ϕ. Consequently, consider

$$\min_{\theta} l^{(\tau)}(\theta) + \frac{\lambda}{2}\|\theta - \phi\|^2$$

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This involves inverting n^2-size matrix, which is infeasible to compute directly. Consider instead

$$\underbrace{\left(I + \frac{H(l^{(\tau)}(\theta))}{\lambda}\right)}_{A} g = \underbrace{\nabla_\theta l^{(\tau)}(\theta)}_{b},$$

which can be solved using conjugate gradient method

A-conjugacy

- Let A be $N \times N$
- We say p A-conjugate with q if $p^\top A q = 0$. And denote $p \perp_A q$

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- If we can keep generating conjugate directions p_k, we can find the solution x that satisfies $Ax = b$ by simply computing $\alpha_i = \frac{\langle p_i, b - Ax_0 \rangle}{\langle p_i A p_i \rangle}$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 \quad (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 \quad (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 \quad (3)\\
& & \cdots & &
\end{aligned}
$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space

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$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$

$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \ = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$.

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \quad p_0 = -r_0, \quad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \quad p_1 = -r_1 + \beta_0 p_0, \quad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \quad p_2 = -r_2 + \beta_1 p_1, \quad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \ = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
    - For $k = 0$, $p_1 = -r_1 + \beta_0 p_0$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \ = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $p_1 = b - Ax_1 + \beta_0 p_0$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$

$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$

$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$

$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \;\; = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = b - Ax_0 - \alpha_0 A p_0 + \beta_0 p_0$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \quad p_0 = -r_0, \quad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \quad p_1 = -r_1 + \beta_0 p_0, \quad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \quad p_2 = -r_2 + \beta_1 p_1, \quad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \ = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = p_0 - \alpha_0 A p_0 + \beta_0 p_0$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \quad p_0 = -r_0, \quad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \quad p_1 = -r_1 + \beta_0 p_0, \quad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \quad p_2 = -r_2 + \beta_1 p_1, \quad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \; = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = p_0 - \alpha_0 Ap_0 + \beta_0 p_0 \;\Rightarrow Ap_0 = \frac{1}{\alpha_0}[p_0 + \beta_0 p_0 - p_1] \in \text{span}\{p_0, p_1\}$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 \quad (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 \quad (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 \quad (3)\\
& & \cdots & &
\end{aligned}
$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \ = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = p_0 - \alpha_0 A p_0 + \beta_0 p_0 \ \Rightarrow A p_0 = \frac{1}{\alpha_0}[p_0 + \beta_0 p_0 - p_1] \in \text{span}\{p_0, p_1\}$
  - Assume $Ap_{k-1} \in \text{span}\{p_0, p_1, \cdots, p_k\}$,

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 \quad (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 \quad (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 \quad (3)\\
&& \cdots &&&
\end{aligned}
$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \; = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = p_0 - \alpha_0 A p_0 + \beta_0 p_0 \;\Rightarrow A p_0 = \frac{1}{\alpha_0}[p_0 + \beta_0 p_0 - p_1] \in \text{span}\{p_0, p_1\}$
  - Assume $Ap_{k-1} \in \text{span}\{p_0, p_1, \cdots, p_k\}$, $p_{k+1} = -r_{k+1} + \beta_k p_k$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 \quad (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 \quad (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 \quad (3)\\
& & \cdots & &
\end{aligned}
$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $p_1 = p_0 - \alpha_0 Ap_0 + \beta_0 p_0 \Rightarrow Ap_0 = \frac{1}{\alpha_0}[p_0 + \beta_0 p_0 - p_1] \in \text{span}\{p_0, p_1\}$
  - Assume $Ap_{k-1} \in \text{span}\{p_0, p_1, \cdots, p_k\}$, $p_{k+1} = p_0 - \sum_{i=0}^{k-1} \alpha_i Ap_i - \alpha_k Ap_k + \beta_k p_k$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 & (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 & (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 & (3)\\
& & \cdots & & &
\end{aligned}
$$

- From the middle equations, $\{p_0, p_1, \cdots, p_n\}$ and $\{r_0, r_1, \cdots, r_n\}$ span the same space
- $r_k \quad = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$.
- $Ap_k \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$ for $k \geq 0$. We can show that with induction
  - For $k = 0$, $\quad p_1 = p_0 - \alpha_0 Ap_0 + \beta_0 p_0 \Rightarrow Ap_0 = \frac{1}{\alpha_0}[p_0 + \beta_0 p_0 - p_1] \in \text{span}\{p_0, p_1\}$
  - Assume $Ap_{k-1} \in \text{span}\{p_0, p_1, \cdots, p_k\}$, $p_{k+1} = p_0 - \sum_{i=0}^{k-1} \alpha_i Ap_i - \alpha_k Ap_k + \beta_k p_k$ $\Rightarrow Ap_k = \frac{1}{\alpha_k}[p_0 - \sum_{i=0}^{k-1} \alpha_i Ap_i + \beta_k p_k - p_{k+1}] \in \text{span}\{p_0, p_1, \cdots, p_{k+1}\}$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$ and $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$
\begin{aligned}
r_0 &= Ax_0 - b, & p_0 &= -r_0, & x_1 &= x_0 + \alpha_0 p_0 \quad (1)\\
r_1 &= Ax_1 - b, & p_1 &= -r_1 + \beta_0 p_0, & x_2 &= x_1 + \alpha_1 p_1 \quad (2)\\
r_2 &= Ax_2 - b, & p_2 &= -r_2 + \beta_1 p_1, & x_3 &= x_2 + \alpha_2 p_2 \quad (3)\\
& & \cdots & &
\end{aligned}
$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$ and $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$

- From the choice of α_n, we see that $r_n \to 0$ and $x_n \to x$ as long as $p_i \perp_A p_j, i \neq j$

Conjugate gradient method

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$ and $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$

- From the choice of α_n, we see that $r_n \to 0$ and $x_n \to x$ as long as $p_i \perp_A p_j, i \neq j$
- We will show that in the next several slides with induction, note that we also have $p_i \perp r_j$ for $i < j$. It is convenient to show them together

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0\rangle}{\langle p_n A p_n\rangle} = \frac{\langle p_n, p_0\rangle}{\langle p_n A p_n\rangle}$, $\beta_n = \frac{\langle r_{n+1} A p_n\rangle}{\langle p_n A p_n\rangle}$ and $r_k = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$

- From the choice of β, we have $\langle p_{k+1} A p_k\rangle = \langle -r_{k+1} + \beta_k p_k, Ap_k\rangle$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$

$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$

$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$

$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$, $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$ and $r_k = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$

- From the choice of β, we have $\langle p_{k+1} A p_k \rangle = \langle -r_{k+1} + \frac{\langle r_{k+1} A p_k \rangle}{\langle p_k A p_k \rangle} p_k, A p_k \rangle = 0$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \quad p_0 = -r_0, \quad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \quad p_1 = -r_1 + \beta_0 p_0, \quad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \quad p_2 = -r_2 + \beta_1 p_1, \quad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0\rangle}{\langle p_n A p_n\rangle} = \frac{\langle p_n, p_0\rangle}{\langle p_n A p_n\rangle}$, $\beta_n = \frac{\langle r_{n+1} A p_n\rangle}{\langle p_n A p_n\rangle}$ and $r_k = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$

- From the choice of β, we have $\langle p_{k+1} A p_k\rangle = \langle -r_{k+1} + p_k, Ap_k\rangle = 0 \Rightarrow p_{k+1} \perp_A p_k$. In particular, $p_1 \perp_A p_0$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$, $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$ and $r_k = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$

- From the choice of β, we have $\langle p_{k+1} A p_k \rangle = \langle -r_{k+1} + p_k, A p_k \rangle = 0 \Rightarrow p_{k+1} \perp_A p_k$. In particular, $p_1 \perp_A p_0$
- And $\langle p_0, r_1 \rangle = \langle p_0, p_0 - \alpha_0 A p_0 \rangle$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \quad p_0 = -r_0, \quad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \quad p_1 = -r_1 + \beta_0 p_0, \quad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \quad p_2 = -r_2 + \beta_1 p_1, \quad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n A p_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n A p_n \rangle}$, $\beta_n = \frac{\langle r_{n+1} A p_n \rangle}{\langle p_n A p_n \rangle}$ and $r_k = -p_0 + \alpha_0 A p_0 + \alpha_1 A p_1 + \cdots + \alpha_{k-1} A p_{k-1}$

- From the choice of β, we have $\langle p_{k+1} A p_k \rangle = \langle -r_{k+1} + p_k, Ap_k \rangle = 0 \Rightarrow p_{k+1} \perp_A p_k$. In particular, $p_1 \perp_A p_0$
- And $\langle p_0, r_1 \rangle = \langle p_0, p_0 - \frac{\langle p_0, p_0 \rangle}{\langle p_0 A p_0 \rangle} A p_0 \rangle$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$

$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$

$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$

$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n Ap_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n Ap_n \rangle}$, $\beta_n = \frac{\langle r_{n+1} Ap_n \rangle}{\langle p_n Ap_n \rangle}$ and $r_k = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$

- From the choice of β, we have $\langle p_{k+1} Ap_k \rangle = \langle -r_{k+1} + p_k, Ap_k \rangle = 0 \Rightarrow p_{k+1} \perp_A p_k$. In particular, $p_1 \perp_A p_0$
- And $\langle p_0, r_1 \rangle = \langle p_0, p_0 - \frac{\langle p_0, p_0 \rangle}{\langle p_0 Ap_0 \rangle} Ap_0 \rangle = 0$. Thus, $p_0 \perp r_1$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

where $\alpha_n = \frac{\langle p_n, b - Ax_0 \rangle}{\langle p_n, Ap_n \rangle} = \frac{\langle p_n, p_0 \rangle}{\langle p_n, Ap_n \rangle}$, $\beta_n = \frac{\langle r_{n+1}, Ap_n \rangle}{\langle p_n, Ap_n \rangle}$ and $r_k = -p_0 + \alpha_0 Ap_0 + \alpha_1 Ap_1 + \cdots + \alpha_{k-1} Ap_{k-1}$

- For $i < j \le k$, assume that we have $p_i \perp_A p_j$ and $p_i \perp r_j$

Conjugacy proof

Consider an initial guess x_0 for the problem $Ax = b$. And iterate as follow

$$r_0 = Ax_0 - b, \qquad p_0 = -r_0, \qquad x_1 = x_0 + \alpha_0 p_0 \tag{1}$$
$$r_1 = Ax_1 - b, \qquad p_1 = -r_1 + \beta_0 p_0, \qquad x_2 = x_1 + \alpha_1 p_1 \tag{2}$$
$$r_2 = Ax_2 - b, \qquad p_2 = -r_2 + \beta_1 p_1, \qquad x_3 = x_2 + \alpha_2 p_2 \tag{3}$$
$$\cdots$$

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  - Thus, by induction, $p_i \perp_A p_j$ (and $p_i \perp r_j$) for all $i < j$

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- There is no need to actually compute the Hessian, the update only involves Hessian-vector product Hv, we can compute

$$Hv = \left.\frac{d}{dt}\right|_{t=0} \nabla f(x + tv)$$

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  - Model-based approach: design network that avoids overfitting

Reference

- Hung-yi Lee's MAML lecture
- An Interactive Introduction to Model-Agnostic Meta-Learning