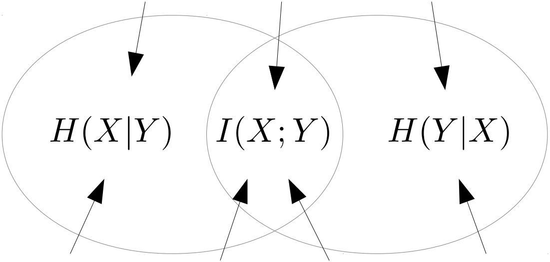
Review

$$H(X,Y) = H(X|Y) + I(X;Y) + H(Y|X)$$



$$H(X) = H(X|Y) + I(X;Y) \qquad I(X;Y) + H(Y|X) = H(Y)$$

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This time

- Identification/Decision trees
- Random forests
- Law of Large Number
- Asymptotic equipartition (AEP) and typical sequences

Vampire database

Romanian Data Base

Vampire?	Shadow?	Garlic?	Complexion?	Accent?
No	?	Yes	Pale	None
No	Yes	Yes	Ruddy	None
Yes	?	No	Ruddy	None
Yes	No	No	Average	Heavy
Yes	?	No	Average	Odd
No	Yes	No	Pale	Heavy
No	Yes	No	Average	Heavy
No	?	Yes	Ruddy	Odd

(https://www.youtube.com/watch?v=SXBG3RGr_Rc)

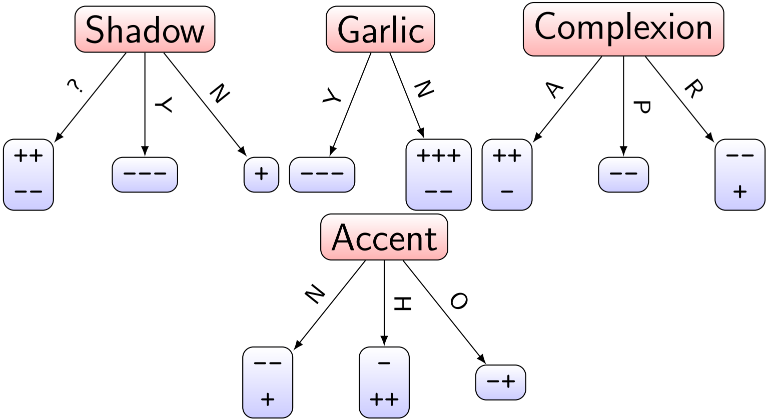
Identifying vampire

Goal: Design a set of tests to identify vampires

Potential difficulties

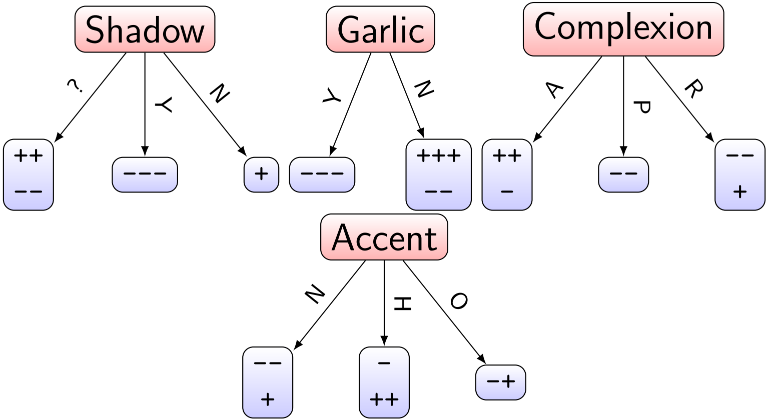
- Non-numerical data
- Some information may not matter
- Some may matter only sometimes
- Tests may be costly $\Rightarrow$ conduct as few as possible

Test trees



$+$: Vampire $\quad -$: Not vampire

Test trees



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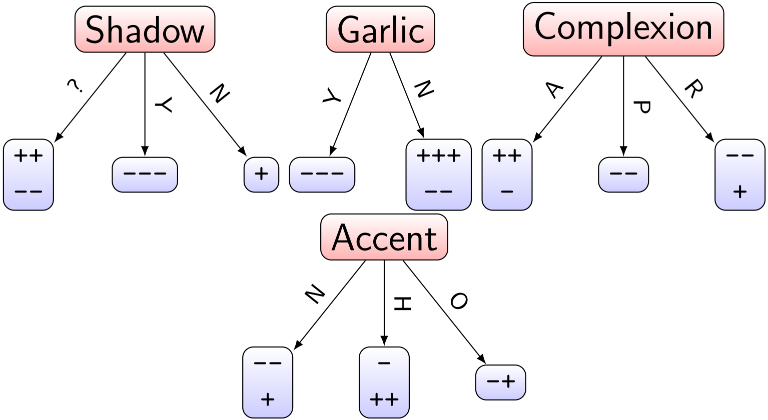
How to pick a good test?

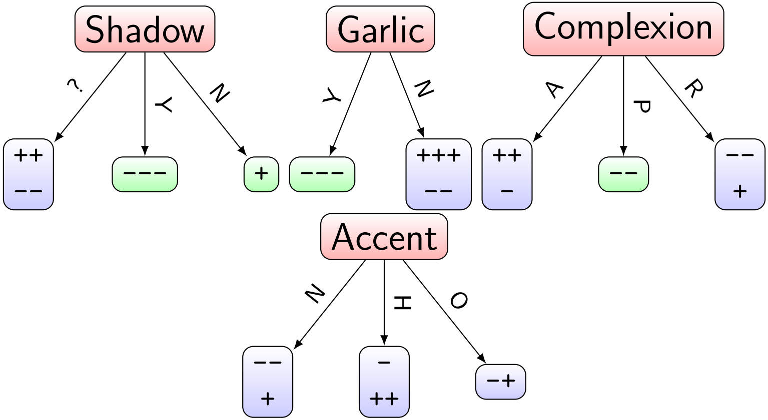
Test trees



$+$: Vampire $-$: Not vampire

How to pick a good test? Pick test that identifies most vampires (and non-vampires)!

Sizes of homogeneous sets



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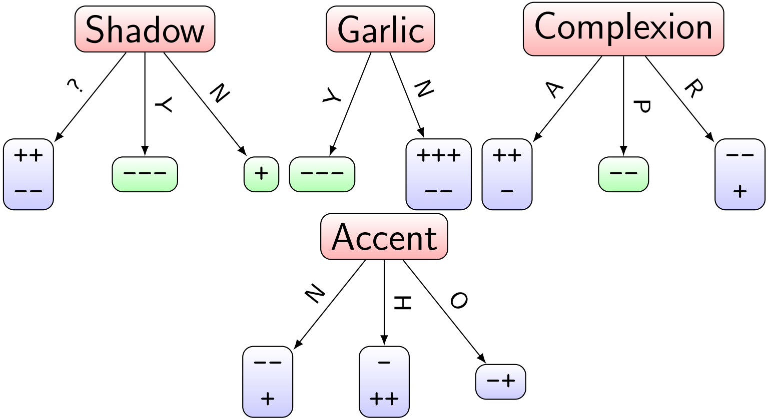
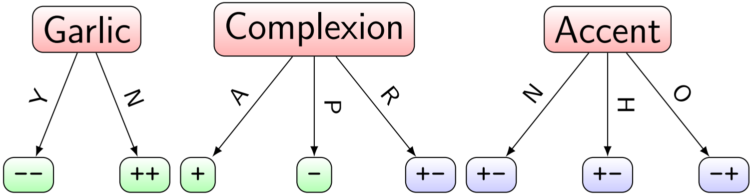


$+$: Vampire $-$: Not vampire

Shadow: 4 Garlic: 3 Complexion: 2 Accent: 0

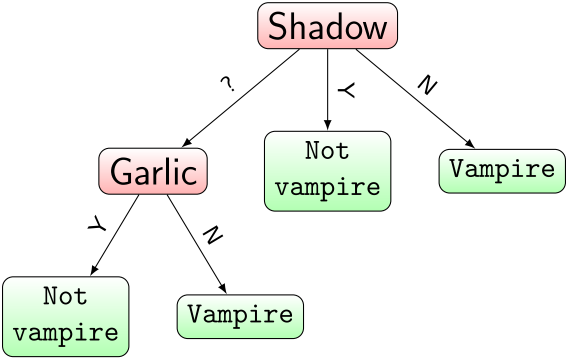
Picking second test

Let say we pick "shadow" as the first test after all. Then, for the remaining unclassified individuals,



Garlic: 4 Complexion: 2 Accent: 0

Combined tests



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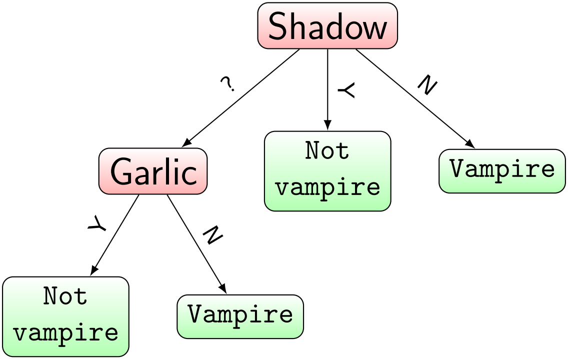

Problem

When our database size increases, none of the test likely to completely separate vampire from non-vampire. All tests will score 0 then.

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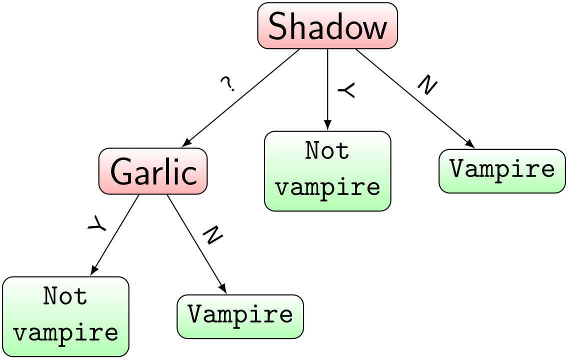

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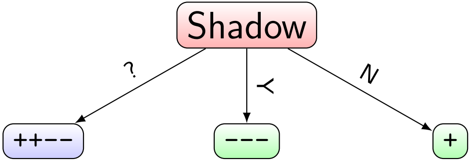
Entropy comes to the rescue!

Conditional entropy as a measure of test efficiency

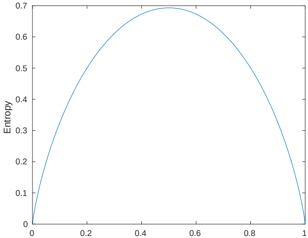
Consider the database is randomly sampled from a distribution. A set is

- Very homogeneous $\approx$ high certainty
- Not so homogenous $\approx$ high randomness

These can be measured with its entropy



$H(V|S=?)=1 \quad H(V|S=Y)=0 \quad H(V|S=N)=0$



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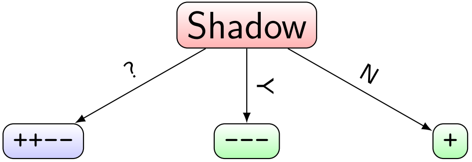

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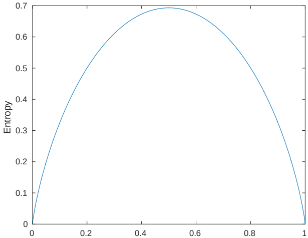

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)$$

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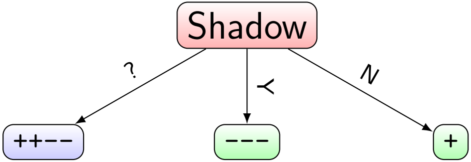

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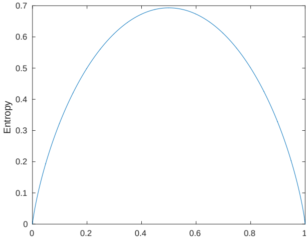

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)$$

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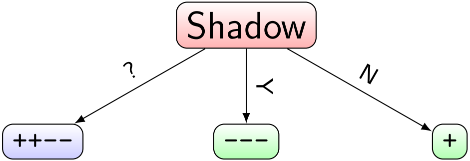

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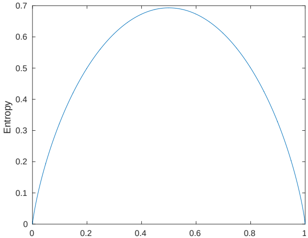

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)+\frac{1}{8}H(V|S=N)=0.5$$

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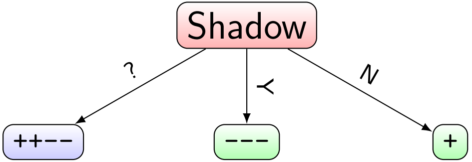

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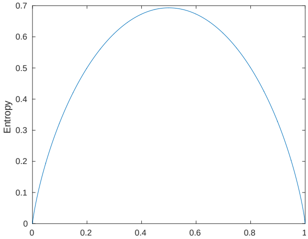

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$$=Pr(S=?)H(V|S=?)+Pr(S=Y)H(V|S=Y)+Pr(S=N)H(V|S=N)$$

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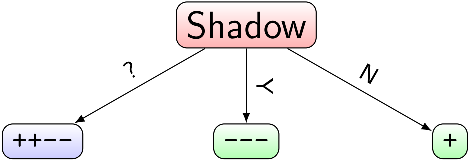

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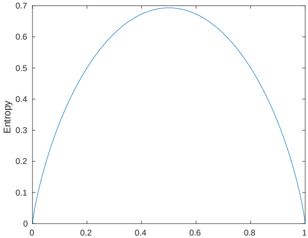

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$$
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=&Pr(S=?)H(V|S=?) + Pr(S=Y)H(V|S=Y) + Pr(S=N)H(V|S=N)\\
=&H(V|S)
\end{aligned}
$$

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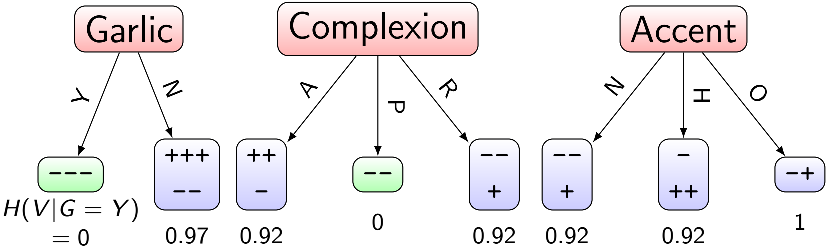

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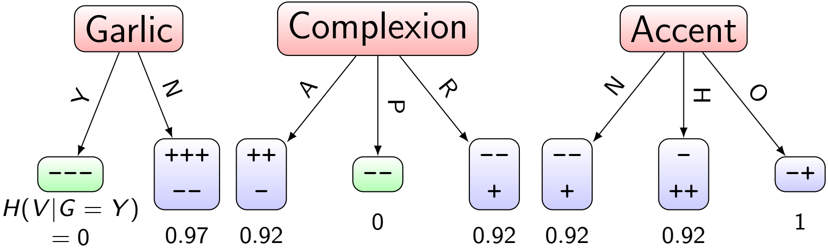

$$H(V|S) = 0.5$$

$$H(V|G) = \frac{3}{8}\cdot 0 + \frac{5}{8}\cdot 0.97 = 0.61$$

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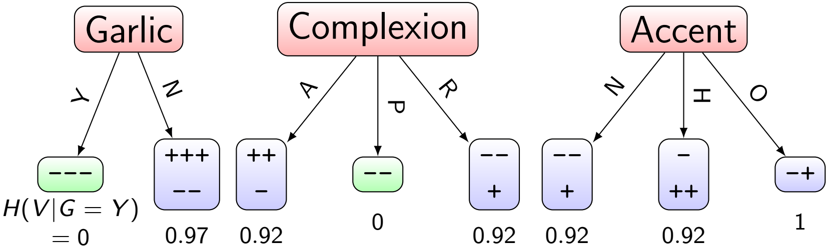

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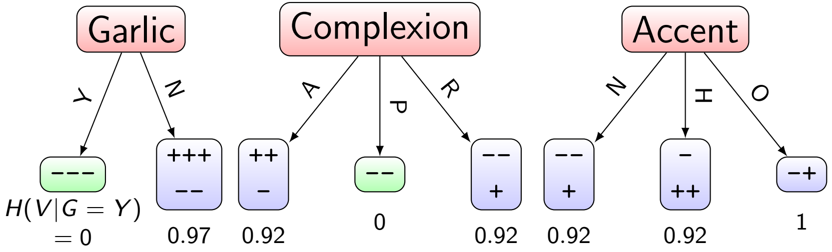

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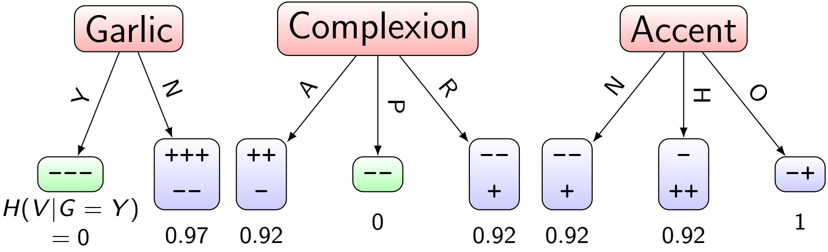

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Order of tests to pick: $S \succ G \succ C \succ A$

Potential extensions

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- Build a number of trees instead of a single tree $\Rightarrow$ random forests

Random forests

- Pick random subset of training samples
- Train on each random subset but limited to a subset of features/attributes
- Given a test sample
  - Classify sample using each of the trees
  - Make final decision based on majority vote

Law of Large Number (LLN)

If we randomly sample $x_1, x_2, \cdots, x_N$ from an i.i.d. (identical and independently distributed) source, the average of $f(x_i)$ will approach the expected value as $N \to \infty$. That is,

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Example

This is precisely how poll supposes to work! Pollster randomly draws sample from a portion of the population but will expect the prediction matches the outcome

Proof of LLN

The LLN is a rather strong result. We will only show a weak version here

$$Pr\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(X_i) - E[f(X)]\right| \geq a\right) \leq \frac{Var(f(X))}{Na^2} \propto \frac{1}{N}$$

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$$Pr(X \geq b) \leq \frac{E[X]}{b} \qquad \text{if } X \geq 0$$

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$X = I(X \geq b) \cdot X + I(X < b) \cdot X \geq I(X \geq b) \cdot b$

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$X = I(X \geq b) \cdot X + I(X < b) \cdot X \geq I(X \geq b) \cdot b \Rightarrow \quad E[X] \geq Pr(X \geq b) \cdot b$

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$$Pr(|Y - E[Y]| \geq a) = Pr(|Y - E[Y]|^2 \geq a^2)$$

$$\leq \frac{E[|Y - E[Y]|^2]}{a^2}$$

Proof of LLN

Markov's Inequality

$$Pr(X \geq b) \leq \frac{E[X]}{b} \qquad \text{if } X \geq 0$$

Chebyshev's Inequality

$$Pr(|Y - E[Y]| \geq a) \leq \frac{Var(Y)}{a^2}$$

Proof: Take $X = |Y - E[Y]|^2$ and $b = a^2$, by Markov's Inequality

$$Pr(|Y - E[Y]| \geq a) = Pr(|Y - E[Y]|^2 \geq a^2)$$

$$\leq \frac{E[|Y - E[Y]|^2]}{a^2} = \frac{Var(Y)}{a^2}$$

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Proof of weak LLN

Let $Z = \frac{1}{N}\sum_{i=1}^{N} f(X_i)$, apparently $E[Z] = E[f(X)]$ and

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Main idea

Consider a sequence of symbols $x_1, x_2, \cdots, x_N$ sampled from a DMS and consider the sample average of the log-probabilities of each sampled symbols

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Rearranging the terms, this implies that for any sequence sampled from the source, the probability of the sampled sequence $p(x^N) \to 2^{-NH(X)}$!

Set of typical sequences

Let's name the sequence x^N with $p(x^N) \sim 2^{-NH(X)}$ typical and define the set of typical sequences

$$\mathcal{A}_\epsilon^N(X) = \{x^N | 2^{-N(H(X)+\epsilon)} \leq p(x^N) \leq 2^{-N(H(X)-\epsilon)}\}$$

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- For any $\epsilon > 0$, we can find a sufficiently large N such that any sampled sequence from the source is typical
- Since all typical sequences have probability $\sim 2^{-NH(X)}$ and they fill up the entire probability space (everything is typical), there should be approximately $\frac{1}{2^{-NH(X)}} = 2^{NH(X)}$ typical sequences

Precise bounds on the size of typical set

$$(1-\delta)2^{N(H(X)-\epsilon)} \leq |\mathcal{A}_\epsilon^N(X)| \leq 2^{N(H(X)+\epsilon)}$$

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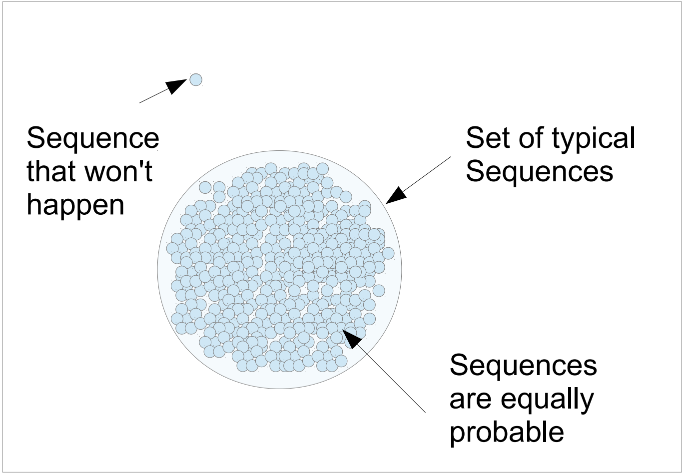
AEP



Asymptotic equipatition refers to the fact that the probability space is equally partitioned by the typical sequences

AEP and compression limit

Consider coin flipping again, let say $Pr(Head) = 0.3$ and $N = 1000$

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- AEP also tells us that the number of typical sequences are approximately $2^{NH(X)}$
- Therefore, we can simply assign index to all the typical sequences and ignore the rest. Then we only need $\log 2^{NH(X)} = NH(X)$ to store a sequence of N symbols. And on average, we need $H(X)$ bits per symbol as before!