# Previously...

- Identification/Decision trees
- Random forests
- Law of Large Number
- Asymptotic equipartition (AEP) and typical sequences

# This time

- Joint typical sequences
- Covering and Packing Lemmas
- Channel coding setup
- Channel coding rate
- Channel capacity
- Channel Coding Theorem

# Jointly typical sequences

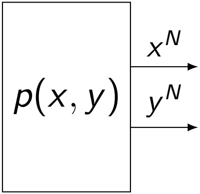
For a pair of sequences $x^N$ and $y^N$, we say that they are jointly typical if

$$2^{-N(H(X,Y)+\epsilon)} \leq p(x^N, y^N) \leq 2^{-N(H(X,Y)-\epsilon)}$$

and $x^N$ and $y^N$ themselves are typical

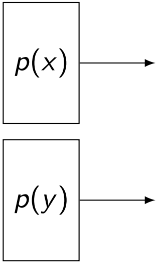
As in the single sequence case,

- Any sequence pair drawing from a joint source $p(x,y)$ is essentially jointly typical
- There are $\sim 2^{NH(X,Y)}$ jointly typical sequences



# Joint typicality of independent seqences

- Given sequences $X^N$ and $Y^N$ independently drawn from discrete memoryless sources $p(x)$ and $p(y)$



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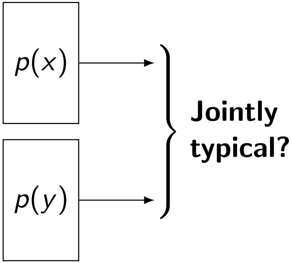

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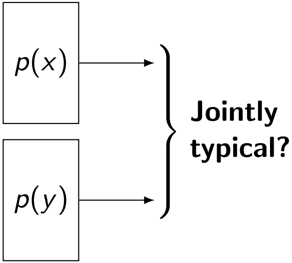

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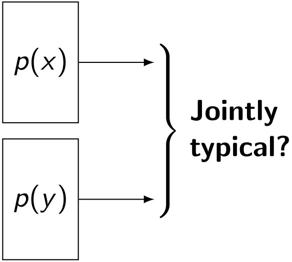

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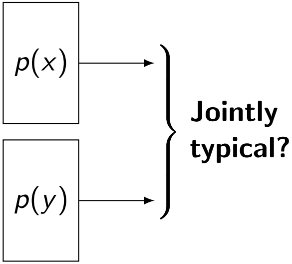

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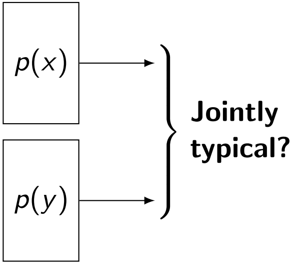

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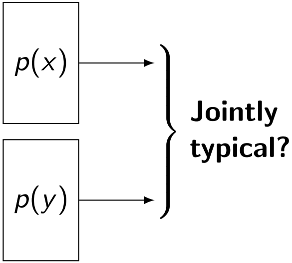

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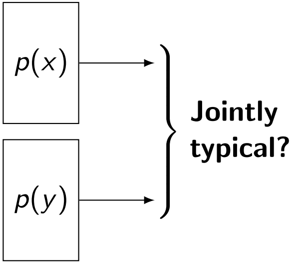

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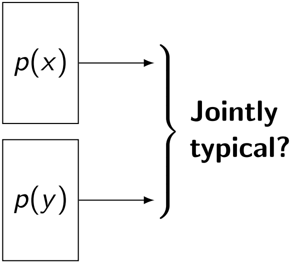

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Since $\epsilon$ can be made arbitrarily small as $N$ increases, as long as $I(X;Y) > R$, we can find a sufficiently large $N$ so that we can "pack" the $M$ $Y^N$ with $X^N$ and none of the $Y^N$ will be jointly typical with $X^N$

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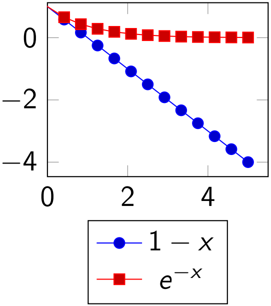

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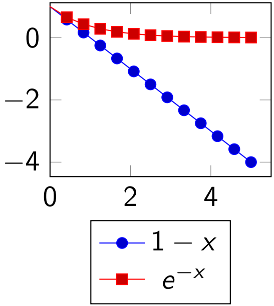


# Summary of packing lemma and covering lemma

## Packing Lemma

We can "pack" $M = 2^{NR}$ (with $R < I(X;Y)$) $x^N$ together without being jointly typical with $y^N$

## Covering Lemma

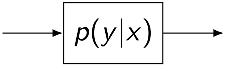
We can "cover" with $M = 2^{NR}$ (with $R > I(X;Y)$) $x^N$ such that at least one $x^N$ being jointly typical with $y^N$

## Remark

- Packing lemma is useful in the proof of channel coding theorem
- Covering lemma is useful in the proof of rate-distortion theorem

We will look into the above applications later in this course

# Channel coding setup

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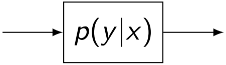


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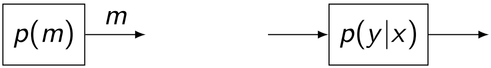


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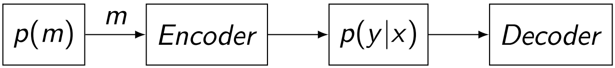


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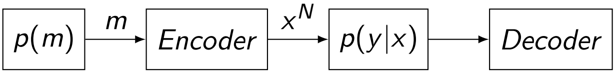


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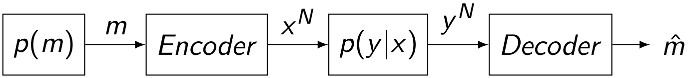


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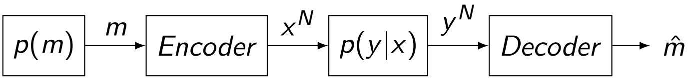
  - Decoder will try to extracted the message from the channel output $y^N$

# Channel coding rate

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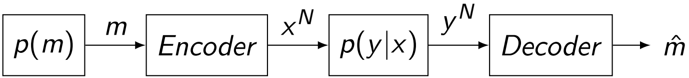


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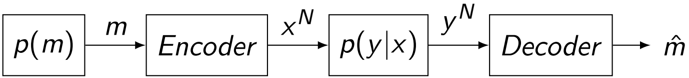


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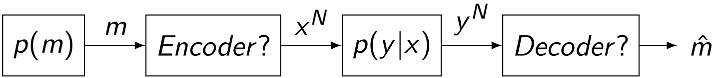
- On the other hand, if $R$ is larger than the capacity $C$, no matter how we try, it is impossible to recontruct $m$ error free
- An intuitive interpretation is that the amount of information can be passed through a channel is just mutual information between the input and output. And since we can pick the statistics of our input, we may make our choice wisely and maximize the mutual information. And the maximum that we can attain is the capacity

# Continuous channel



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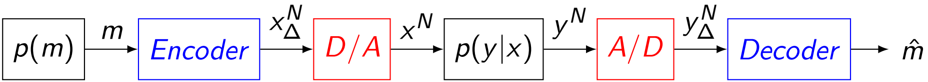


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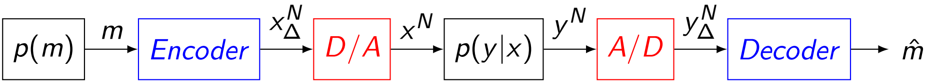


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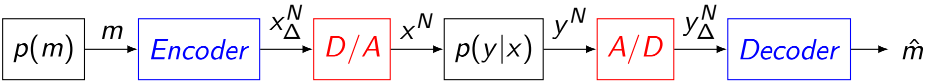


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- As $\Delta \to 0$, $C = \max_{p(x)} I(X;Y)$. So expression is completely the same as the discrete case

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where $SNR$ is the signal to noise ratio

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$$C = 2W\frac{1}{2}\log(1+SNR) = W\log\left(1+\frac{P}{WN_0}\right)$$

# Codebook construction

## Forward statement

If the code rate $R < C = \max_{p(x)} I(X;Y)$, according to the Channel Coding Theorem, we should be able to find a code with encoding mapping $\mathbf{c} : m \in \{1, 2, \cdots, 2^{NR}\} \rightarrow \{0, 1\}^N$ and the error probability of transmitting any message $m \in \{1, 2, \cdots, 2^{NR}\}$, $p_e(m)$, is arbitrarily small

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- The main tool of the proof is random coding
- Let $p^*(x) = \arg\max_{p(x)} I(X;Y)$. Generate codewords from the DMS $p^*(x)$ by sampling $2^n$ length-$n$ sequences from the source:

$$
\begin{aligned}
\mathbf{c}(1) =&(x_1(1), x_2(1), \cdots, x_N(1))\\
\mathbf{c}(2) =&(x_1(2), x_2(2), \cdots, x_N(2))\\
&\cdots\\
\mathbf{c}(2^{NR}) =&(x_1(2^{NR}), x_2(2^{NR}), \cdots, x_N(2^{NR}))
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## Decoding

Upon receiving sequence $\mathbf{y} = (y_1, y_2, \cdots, y_N)$, pick the sequence $\mathbf{c}(m)$ from $\{\mathbf{c}(1), \cdots, \mathbf{c}(2^{NR})\}$ such that $(\mathbf{c}(m), \mathbf{y})$ are jointly typical. That is $p_{X^N,Y^N}(\mathbf{c}(m), \mathbf{y}) \sim 2^{-nH(X,Y)}$. If no such $\mathbf{c}(m)$ exists or more than one such sequence exist, announce error. Otherwise output the decoded message as $m$

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Since $\epsilon$ can be made arbitrarily small as $N$ increase, as long as $I(X; Y) > R$, we can make $P_2$ arbitrarily small also given a sufficiently large $N$

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- Even though the rate reduces from $N$ to $R - \frac{1}{N}$ (number of messages from $2^{NR} \to 2^{NR-1}$). But we can still make the final rate arbitrarily close to the capacity as $N \to \infty$