Previously...

- Joint typical sequences
- Covering and Packing Lemmas
- Channel Coding Theorem
- Capacity of Gaussian channel
- Capacity of additive white Gaussian channel
- Forward proof of Channel Coding Theorem

This time

- Converse Proof of Channel Coding Theorem
- Non-white Gaussian Channel
- Rate-distortion problems
- Rate-distortion Theorem

Converse proof

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To continue the converse proof, we will need to introduce a simple result from Fano

Fano's inequality

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Denote $Pr(error) = P_e = Pr(M \neq \hat{M})$, then $H(M|Y^N) \leq 1 + P_e H(M)$
Intuitively, if $P_e \to 0$, on average we will know M for certain given y and thus $\frac{1}{N} H(M|Y^N) \to 0$

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&\leq 1 + 0 + P_e H(M|Y^N, E=1) \overset{(d)}{\leq} 1 + P_e H(M)
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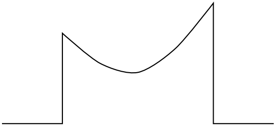
as $N \to \infty$ by Fano's inequality

Color channels

- We look into capacity of white Gaussian channel last time

Color channels



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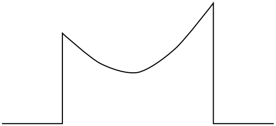

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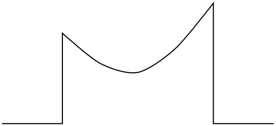

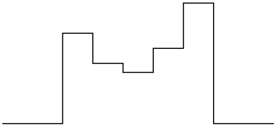

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- Without loss of generality, let's consider the discrete approximation, parallel Gaussian channel

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- Therefore, for the k-th channel, we can transmit $\frac{1}{2}\log\left(1+\frac{P_k}{\sigma_k^2}\right)$ bits per channel use
- So our goal is to assign $P_1, P_2, \cdots, P_K \geq 0$ ($\sum_{k=1}^K P_k \leq P$) such that the total capacity

$$\sum_{k=1}^K \frac{1}{2}\log\left(1+\frac{P_k}{\sigma_k^2}\right)$$

is maximize

KKT conditions

Let's list all the KKT conditions for the optimization problem

$$\max \sum_{k=1}^{K} \frac{1}{2} \log\left(1+\frac{P_k}{\sigma_k^2}\right) \qquad \text{such that}$$

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Capacity of parallel channels

$$\frac{\partial}{\partial P_i}\left[\sum_{k=1}^{K}\frac{1}{2}\log\left(1+\frac{P_k}{\sigma_k^2}\right)+\sum_{k=1}^{K}\lambda_k P_k-\mu\left(\sum_{k=1}^{K}P_k-P\right)\right]=0$$

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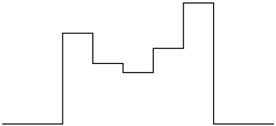
Water-filling interpretation



From $P_i + \sigma_i^2 = const$, power can be allocated intuitively as filling water to a pond (hence "water-filling")

Example

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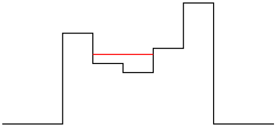

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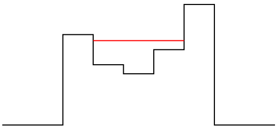

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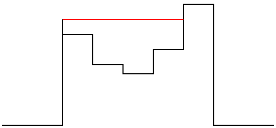

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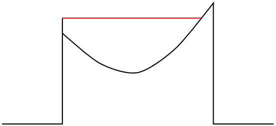

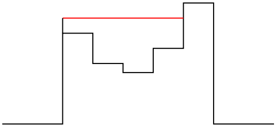

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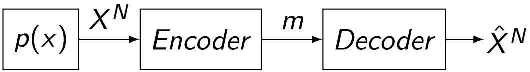
Rate-distortion problem



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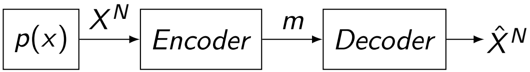

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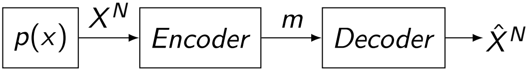

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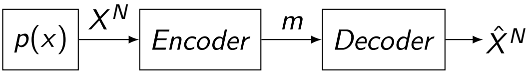
- Let say we are satisfied as long as we can recover X up to certain fidelity, how many bits are needed per sample?

Rate-distortion problem



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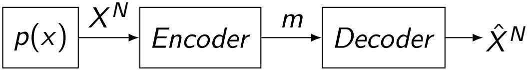
- Let say we are satisfied as long as we can recover X up to certain fidelity, how many bits are needed per sample?
- There is an apparent rate (bits per sample) and distortion (fidelity) trade-off. We expect that needed rate is smaller if we allow a lower fidelity (higher distortion). What we are really interested in is a rate-distortion function

Rate-distortion function

$$m \in \{1, 2, \cdots, M\}$$



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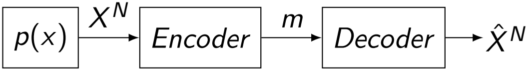

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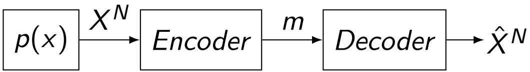

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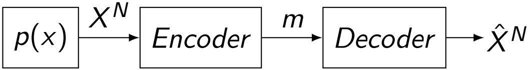

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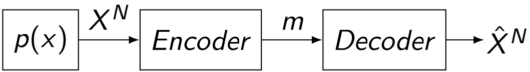

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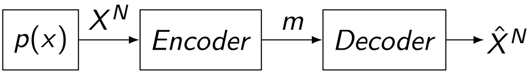

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- Therefore given $\mathcal{D}$, the rate-distortion function is simply

$$R(\mathcal{D}) = min_{p(\hat{x}|x)} I(\hat{X}; X)$$

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- If decoders know nothing, the best bet will be just always decode head (or tail). Then $D=E[d(X,H)]=0.5$

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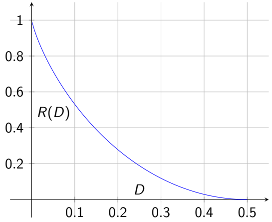

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- Consider $X \sim \mathcal{N}(0, \sigma_X^2)$. To determine the rate-distortion function, we need first to decide the distortion measure. An intuitive will be just the square error. That is,

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Given distortion constraint $\mathcal{D}$, we can find scheme such that the require rate is no bigger than

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Note that the code rate is $\frac{\log 2^{NR}}{N} = R$ as desired

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We say joint typical sequences x^N and $\hat{x}^N$ are distortion typical $((x^N, \hat{x}^N) \in \mathcal{A}^N_{d,\epsilon})$ if $|d(x^N, \hat{x}^N) - E[d(X, \hat{X})]| \leq \epsilon$

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- For two independently drawn sequences $\hat{X}^N$ and X^N, the probability for them to be distortion typical will be just the same as before. In particular, $(1-\delta)2^{-N(I(X;\hat{X})-3\epsilon)} \leq Pr((X^N, \hat{X}^N) \in \mathcal{A}^N_{d,\epsilon}(X, \hat{X}))$

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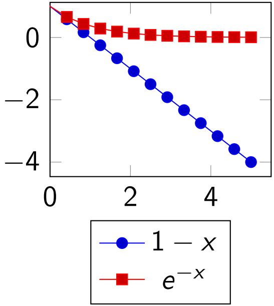

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\le&\exp(-(1-\delta)2^{-N(I(\hat{X};X)-R+3\epsilon)})\to 0 \text{ as } N\to\infty \text{ and } R>I(X;\hat{X})
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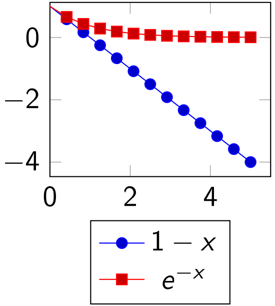

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- By covering Lemma, encoding failure is neglible as long as $R > I(X;\hat{X})$
- If encoding is successful, $\mathbf{C}(i)$ and X^N should be distortion typical. Therefore, $E[d(\mathbf{C}(i);X^N)] \sim E[d(\hat{X},X)] \leq \mathcal{D}$ as desired

Converse proof

Converse statement

If rate is smaller than $R(\mathcal{D})$, distortion will be larger than $\mathcal{D}$

Converse proof

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Alternative statement

If distortion is less than or equal to $\mathcal{D}$, the rate must be larger than $R(\mathcal{D})$

Converse proof

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In the proof, we need to use the convex property of $R(\mathcal{D})$. That is,

$$R(a\mathcal{D}_1 + (1-a)\mathcal{D}_2) \geq aR(\mathcal{D}_1) + (1-a)R(\mathcal{D}_2)$$

So we will digress a little bit to show this convex property first

Log-sum inequality

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For any $a_1, \cdots, a_n \geq 0$ and $b_1, \cdots, b_n \geq 0$, we have

$$\sum_i a_i \log_2 \frac{a_i}{b_i} \geq \sum_i a_i \log_2 \frac{\sum_i a_i}{\sum_i b_i}.$$

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Proof

We can define two distributions $p(x)$ and $q(x)$ with $p(x_i) = \frac{a_i}{\sum_i a_i}$ and $q(x_i) = \frac{b_i}{\sum_i b_i}$. Since $p(x)$ and $q(x)$ are both non-negative and sum up to 1, they are indeed valid probability mass functions.

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$$\begin{aligned} 0 \leq KL(p(x) \| q(x)) &= \sum_i p(x_i) \log_2 \frac{p(x_i)}{q(x_i)} \\ &= \sum_i \frac{a_i}{\sum_i a_i} \left(\log_2 \frac{a_i}{b_i} - \log_2 \frac{\sum_i a_i}{\sum_i b_i} \right) \end{aligned}$$

Convexity of KL-Divergence

For any four distributions $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$, we have

$$\lambda_1 KL(p_1 \| q_1) + \lambda_2 KL(p_2 \| q_2) \geq KL(\lambda_1 p_1 + \lambda_2 p_2 \| \lambda_1 q_1 + \lambda_2 q_2),$$

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\geq&\sum_{x\in\mathcal{X}} (\lambda_1 p_1(x) + \lambda_2 p_2(x))\log\frac{\lambda_1 p_1(x)+\lambda_2 p_2(x)}{\lambda_1 q_1(x)+\lambda_2 q_2(x)} \quad \text{(by log-sum inequality)}
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\end{aligned}$$

Convexity of $I(X;Y)$ with respect to $p(y|x)$

For any random variables X and Y, $I(X;Y)$ is a convex function of $p(y|x)$ for a fixed $p(x)$

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$I(X;Y)$ is concave with respect to $p(x)$ for fixed $p(y|x)$ though. A proof is given in Cover and Thomas and will be omitted here

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Let us write

$$\begin{aligned} I(X;Y) &= KL(p(x,y)\|p(x)p(y)) \\ &= KL\left(p(x)p(y|x)\middle\|p(x)\sum_x p(x)p(y|x)\right) \triangleq f(p(y|x)) \end{aligned}$$

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$$\lambda f(p_1(y|x)) + (1-\lambda)f(p_2(y|x)) \geq f(\lambda p_1(y|x) + (1-\lambda)p_2(y|x))$$

Proof

Continue from previous slide, we have

$$\begin{aligned}
&\lambda f(p_1(y|x)) + (1-\lambda) f(p_2(y|x)) \\
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=& f(\lambda p_1(y|x) + (1-\lambda)p_2(y|x))
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Recall that $R(\mathcal{D}) = \min_{p(\hat{x}|x)} I(\hat{X};X)$ with $E[d(X,\hat{X})] \leq \mathcal{D}$
We want to show that

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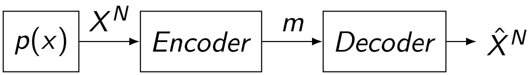
=&I(\tilde{X};X) \geq R(\lambda\mathcal{D}_1 + (1-\lambda)\mathcal{D}_2),
\end{aligned}$$

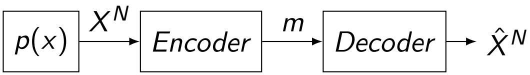
where $\tilde{X} = \begin{cases} \hat{X}_1 & \text{with } \lambda \text{ fraction of time} \\ \hat{X}_2 & \text{with } (1-\lambda) \text{ fraction of time} \end{cases}$

Converse proof



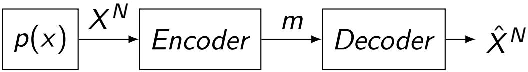
$NR \geq H(M)$

Converse proof



$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M; X^N)$$

Converse proof



$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M; X^N) \geq I(\hat{X}^N; X^N)$$

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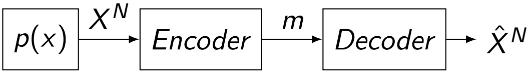
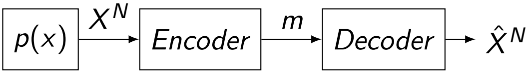
Converse proof



$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N)$$

$$= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}^N, X^{i-1})$$

Converse proof

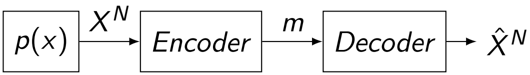


$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M; X^N) \geq I(\hat{X}^N; X^N)$$

$$= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}^N, X^{i-1})$$

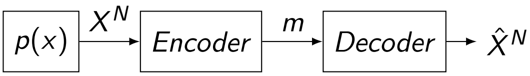
$$\geq \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}_i) = \sum_{i=1}^{N} I(X_i; \hat{X}_i)$$

Converse proof



$$
\begin{aligned}
NR &\geq H(M) \geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N) \\
&= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}^N, X^{i-1}) \\
&\geq \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}_i) = \sum_{i=1}^N I(X_i;\hat{X}_i) \\
&\geq \sum_{i=1}^N R(E[d(X_i,\hat{X}_i)]) = N\left(\frac{1}{N}\sum_{i=1}^N R(E[d(X_i;\hat{X}_i)])\right)
\end{aligned}
$$

Converse proof



$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N)$$

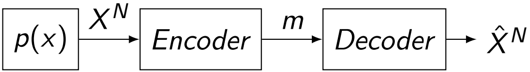
$$= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}^N, X^{i-1})$$

$$\geq \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}_i) = \sum_{i=1}^N I(X_i;\hat{X}_i)$$

$$\geq \sum_{i=1}^N R(E[d(X_i,\hat{X}_i)]) = N\left(\frac{1}{N}\sum_{i=1}^N R(E[d(X_i;\hat{X}_i)])\right)$$

$$\geq NR\left(\frac{1}{N}\sum_{i=1}^N E[d(X_i;\hat{X}_i)]\right)$$

Converse proof



$$NR \geq H(M) \geq H(M) - H(M|X^N) = I(M; X^N) \geq I(\hat{X}^N; X^N)$$

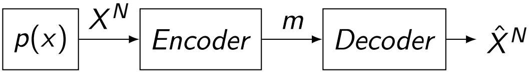
$$= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}^N, X^{i-1})$$

$$\geq \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}_i) = \sum_{i=1}^{N} I(X_i; \hat{X}_i)$$

$$\geq \sum_{i=1}^{N} R(E[d(X_i, \hat{X}_i)]) = N\left(\frac{1}{N}\sum_{i=1}^{N} R(E[d(X_i; \hat{X}_i)])\right)$$

$$\geq NR\left(\frac{1}{N}\sum_{i=1}^{N} E[d(X_i; \hat{X}_i)]\right) = NR\left(E\left[\frac{1}{N}\sum_{i=1}^{N} d(X_i; \hat{X}_i)\right]\right)$$

Converse proof



$$
\begin{aligned}
NR &\geq H(M) \geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N) \\
&= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}^N, X^{i-1}) \\
&\geq \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}_i) = \sum_{i=1}^{N} I(X_i;\hat{X}_i) \\
&\geq \sum_{i=1}^{N} R(E[d(X_i,\hat{X}_i)]) = N\left(\frac{1}{N}\sum_{i=1}^{N} R(E[d(X_i;\hat{X}_i)])\right) \\
&\geq NR\left(\frac{1}{N}\sum_{i=1}^{N} E[d(X_i;\hat{X}_i)]\right) = NR\left(E\left[\frac{1}{N}\sum_{i=1}^{N} d(X_i;\hat{X}_i)\right]\right) \\
&= NR(E[d(X^N;\hat{X}^N)]) \geq NR(D)
\end{aligned}
$$