Previously...

- Converse Proof of Channel Coding Theorem
- Non-white Gaussian Channel
- Rate-distortion problems

This time

- Proof of the Rate-distortion Theorem

Gaussian source

- Consider $X \sim \mathcal{N}(0, \sigma_X^2)$. To determine the rate-distortion function, we need first to decide the distortion measure. An intuitive will be just the square error. That is,

$$d(\hat{X}, X) = (\hat{X} - X)^2$$

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Forward proof

Forward statement

Given distortion constraint $\mathcal{D}$, we can find scheme such that the require rate is no bigger than

$$R(\mathcal{D}) = \min_{p(\hat{x}|x)} I(X;\hat{X}),$$

where the $\hat{X}$ introduced by $p(\hat{x}|x)$ should satisfy $E[d(X,\hat{X})] \leq \mathcal{D}$

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Let say $p^*(\hat{x}|x)$ is the distribution that achieve the rate-distortion optimiation problem. Randomly construct 2^{NR} codewords as follows

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- Repeat this N time to get a length-N codeword
- Store the i-th codeword as $\mathbf{C}(i)$

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Note that the code rate is $\frac{\log 2^{NR}}{N} = R$ as desired

Covering lemma and distortion typical sequences

We say joint typical sequences x^N and $\hat{x}^N$ are distortion typical $((x^N, \hat{x}^N) \in \mathcal{A}^N_{d,\epsilon})$ if $|d(x^N, \hat{x}^N) - E[d(X, \hat{X})]| \leq \epsilon$

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- Consequently, $(1-\delta)2^{N(H(X,\hat{X})-\epsilon)} \le |\mathcal{A}^N_{d,\epsilon}| \le 2^{N(H(X,\hat{X})+\epsilon)}$ as before

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- Consequently, $(1-\delta)2^{N(H(X,\hat{X})-\epsilon)} \leq |\mathcal{A}^N_{d,\epsilon}| \leq 2^{N(H(X,\hat{X})+\epsilon)}$ as before
- For two independently drawn sequences $\hat{X}^N$ and X^N, the probability for them to be distortion typical will be just the same as before. In particular, $(1-\delta)2^{-N(I(X;\hat{X})-3\epsilon)} \leq Pr((X^N, \hat{X}^N) \in \mathcal{A}^N_{d,\epsilon}(X, \hat{X}))$

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\leq&(1-(1-\delta)2^{-N(I(\hat{X};X)+3\epsilon)})^M \\
\leq&\exp(-M(1-\delta)2^{-N(I(\hat{X};X)+3\epsilon)})
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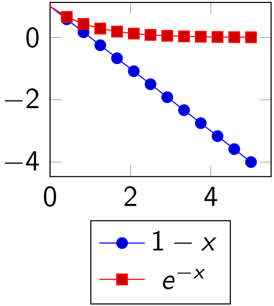
$$



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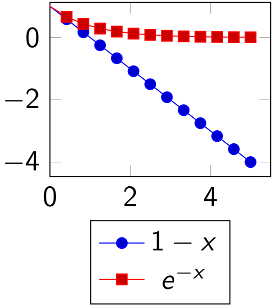
$$\leq(1-(1-\delta)2^{-N(I(\hat{X};X)+3\epsilon)})^M$$
$$\leq \exp(-M(1-\delta)2^{-N(I(\hat{X};X)+3\epsilon)})$$
$$\leq \exp(-(1-\delta)2^{-N(I(\hat{X};X)-R+3\epsilon)})\rightarrow 0 \text{ as } N\rightarrow\infty \text{ and } R>I(X;\hat{X})+3\epsilon$$



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Encoding

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- First of all, the only point of failure lies on encoding, that is when the encoder cannot find a codeword jointly typical with X^N

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Performance analysis

- First of all, the only point of failure lies on encoding, that is when the encoder cannot find a codeword jointly typical with X^N
- By covering Lemma, encoding failure is negligible as long as $R > I(X;\hat{X})$
- If encoding is successful, $\mathbf{C}(i)$ and X^N should be distortion typical. Therefore, $E[d(\mathbf{C}(i);X^N)] \sim E[d(\hat{X},X)] \leq \mathcal{D}$ as desired

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In the proof, we need to use the convex property of $R(\mathcal{D})$. That is,

$$R(a\mathcal{D}_1 + (1-a)\mathcal{D}_2) \geq aR(\mathcal{D}_1) + (1-a)R(\mathcal{D}_2)$$

So we will digress a little bit to show this convex property first

Log-sum inequality

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For any $a_1, \cdots, a_n \geq 0$ and $b_1, \cdots, b_n \geq 0$, we have

$$\sum_i a_i \log_2 \frac{a_i}{b_i} \geq \left(\sum_i a_i \right) \log_2 \frac{\sum_i a_i}{\sum_i b_i}.$$

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Proof

We can define two distributions $p(x)$ and $q(x)$ with $p(x_i) = \frac{a_i}{\sum_i a_i}$ and $q(x_i) = \frac{b_i}{\sum_i b_i}$. Since $p(x)$ and $q(x)$ are both non-negative and sum up to 1, they are indeed valid probability mass functions.

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$$0 \leq KL(p(x)\|q(x)) = \sum_i p(x_i) \log_2 \frac{p(x_i)}{q(x_i)}$$

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$$\begin{aligned} 0 \leq KL(p(x)\|q(x)) &= \sum_i p(x_i) \log_2 \frac{p(x_i)}{q(x_i)} \\ &= \sum_i \frac{a_i}{\sum_i a_i} \left(\log_2 \frac{a_i}{b_i} - \log_2 \frac{\sum_i a_i}{\sum_i b_i}\right) \end{aligned}$$

Convexity of KL-Divergence

For any four distributions $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$, we have

$$\lambda_1 KL(p_1\|q_1) + \lambda_2 KL(p_2\|q_2) \geq KL(\lambda_1 p_1 + \lambda_2 p_2 \| \lambda_1 q_1 + \lambda_2 q_2),$$

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Proof

$$\begin{aligned}&\lambda_1 KL(p_1\|q_1) + \lambda_2 KL(p_2\|q_2)\\ =&\lambda_1 \sum_{x\in\mathcal{X}} p_1(x)\log\frac{p_1(x)}{q_1(x)} + \lambda_2 \sum_{x\in\mathcal{X}} p_2(x)\log\frac{p_2(x)}{q_2(x)}\end{aligned}$$

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\end{aligned}$$

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=&\sum_{x\in\mathcal{X}} \lambda_1 p_1(x) \log \frac{\lambda_1 p_1(x)}{\lambda_1 q_1(x)} + \lambda_2 p_2(x) \log \frac{\lambda_2 p_2(x)}{\lambda_2 q_2(x)} \\
\geq&\sum_{x\in\mathcal{X}} (\lambda_1 p_1(x) + \lambda_2 p_2(x)) \log \frac{\lambda_1 p_1(x) + \lambda_2 p_2(x)}{\lambda_1 q_1(x) + \lambda_2 q_2(x)} \quad \text{(by log-sum inequality)}
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\geq&\sum_{x\in\mathcal{X}} (\lambda_1 p_1(x) + \lambda_2 p_2(x)) \log \frac{\lambda_1 p_1(x) + \lambda_2 p_2(x)}{\lambda_1 q_1(x) + \lambda_2 q_2(x)} \quad \text{(by log-sum inequality)} \\
=&KL(\lambda_1 p_1 + \lambda_2 p_2 \| \lambda_1 q_1 + \lambda_2 q_2)
\end{aligned}$$

Convexity of $I(X;Y)$ with respect to $p(y|x)$

For any random variables X and Y, $I(X;Y)$ is a convex function of $p(y|x)$ for a fixed $p(x)$

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Remark

$I(X;Y)$ is concave with respect to $p(x)$ for fixed $p(y|x)$ though. A proof is given in Cover and Thomas and will be omitted here

Convexity of $I(X;Y)$ with respect to $p(y|x)$

For any random variables X and Y, $I(X;Y)$ is a convex function of $p(y|x)$ for a fixed $p(x)$

Remark

$I(X;Y)$ is concave with respect to $p(x)$ for fixed $p(y|x)$ though. A proof is given in Cover and Thomas and will be omitted here

Proof

Let us write

$$
\begin{aligned}
I(X;Y) &= KL(p(x,y)\|p(x)p(y)) \\
&= KL\left(p(x)p(y|x)\middle\|p(x)\sum_x p(x)p(y|x)\right) \triangleq f(p(y|x))
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We want to show

$$\lambda f(p_1(y|x)) + (1-\lambda)f(p_2(y|x)) \geq f(\lambda p_1(y|x) + (1-\lambda)p_2(y|x))$$

Proof

Continue from previous slide, we have

$$\begin{aligned}
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=&f(\lambda p_1(y|x) + (1-\lambda)p_2(y|x))
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Convexity of $R(\mathcal{D})$

Recall that $R(\mathcal{D}) = \min_{p(\hat{x}|x)} I(\hat{X}; X)$ with $E[d(X, \hat{X})] \leq \mathcal{D}$
We want to show that

$$R(\lambda \mathcal{D}_1 + (1-\lambda)\mathcal{D}_2) \leq \lambda R(\mathcal{D}_1) + (1-\lambda) R(\mathcal{D}_2)$$

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$$\begin{aligned}
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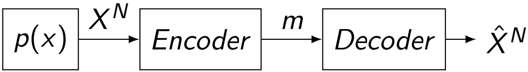
=&\lambda f(p_1^*(\hat{x}|x)) + (1-\lambda) f(p_2^*(\hat{x}|x)) \ge f(\lambda p_1^*(\hat{x}|x) + (1-\lambda) p_2^*(\hat{x}|x)) \\
=&I(\tilde{X}; X) \ge R(\lambda\mathcal{D}_1 + (1-\lambda)\mathcal{D}_2),
\end{aligned}$$

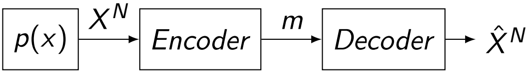
where $\tilde{X} = \begin{cases} \hat{X}_1 & \text{with } \lambda \text{ fraction of time} \\ \hat{X}_2 & \text{with } (1-\lambda) \text{ fraction of time} \end{cases}$

Converse proof



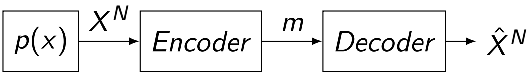
$H(M)$

Converse proof



$$H(M) \geq H(M) - H(M|X^N) = I(M; X^N)$$

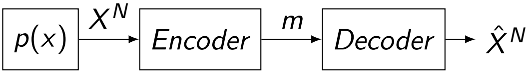
Converse proof



$$H(M) \geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N)$$

$$= H(X^N) - H(X^N|\hat{X}^N)$$

Converse proof



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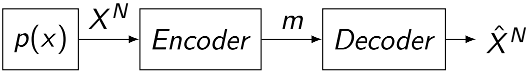
$$= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^{N} H(X_i) - \sum_{i=1}^{N} H(X_i|\hat{X}^N, X^{i-1})$$

Converse proof



$$
\begin{aligned}
H(M) &\geq H(M) - H(M|X^N) = I(M;X^N) \geq I(\hat{X}^N;X^N) \\
&= H(X^N) - H(X^N|\hat{X}^N) = \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}^N, X^{i-1}) \\
&\geq \sum_{i=1}^N H(X_i) - \sum_{i=1}^N H(X_i|\hat{X}_i) = \sum_{i=1}^N I(X_i;\hat{X}_i)
\end{aligned}
$$

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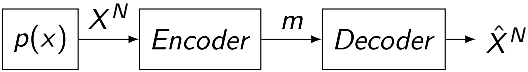

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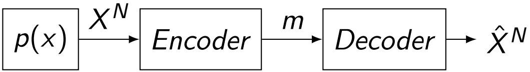
&\geq \sum_{i=1}^{N} R(E[d(X_i,\hat{X}_i)]) = N\left(\frac{1}{N}\sum_{i=1}^{N} R(E[d(X_i;\hat{X}_i)])\right)
\end{aligned}
$$

Converse proof



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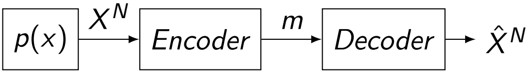
&\geq NR\left(\frac{1}{N}\sum_{i=1}^{N} E[d(X_i;\hat{X}_i)]\right)
\end{aligned}
$$

Converse proof



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$$\geq NR\left(\frac{1}{N}\sum_{i=1}^{N} E[d(X_i; \hat{X}_i)]\right) = NR\left(E\left[\frac{1}{N}\sum_{i=1}^{N} d(X_i; \hat{X}_i)\right]\right)$$

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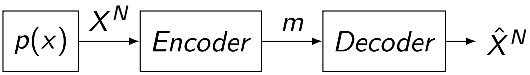

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&= NR(E[d(X^N; \hat{X}^N)]) \geq NR(D)
\end{aligned}
$$