Review

- Univariate Normal: $\mathcal{N}(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- Multivariate Normal: $\mathcal{N}(\mathbf{x};\boldsymbol{\mu},\Sigma) = \frac{1}{det(2\pi\Sigma)} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}$

Remark

Note that $\mathcal{N}(\mathbf{x};\boldsymbol{\mu},\Sigma) = \mathcal{N}(\boldsymbol{\mu};\mathbf{x},\Sigma)$. It is trivial but quite useful

Symmetric matrices

Lemma

$$(M^T)^{-1} = (M^{-1})^T$$

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Proof.

$(M^{-1})^T M^T = (MM^{-1})^T = I \Rightarrow (M^{-1})^T$ is inverse of M^T □

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Proof.

$(M^{-1})^T = (M^T)^{-1} = M^{-1}$ □

Hermitian matrices

- An extension of transpose operation to complex matrices is the hermitian transpose operation, which is simply the transpose and conjugate of a matrix (vector)
- We denote the hermitian transpose of M as $M^\dagger \triangleq \overline{M}^T$, when $\overline{M}$ is the complex conjugate of M
- A matrix is Hermitian if $M^\dagger = M$. Note that a real symmetric matrix is Hermitian

Eigenvalues of Hermitian matrices

Lemma

If M is Hermitian ($M^\dagger = M$), all eigenvalues are real

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Proof.

$$\overline{\lambda}(x^\dagger x) = (\lambda x)^\dagger x = (Mx)^\dagger x = x^\dagger M^\dagger x = x^\dagger M x = x^\dagger(\lambda x) = \lambda(x^\dagger x)$$ □

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If M is Hermitian, eigenvectors of different eigenvalues are orthogonal

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Lemma

If M is Hermitian, eigenvectors of different eigenvalues are orthogonal

Proof.

$$\lambda_1 x_1^\dagger x_2 = (Mx_1)^\dagger x_2 = x_1^\dagger M x_2 = \lambda_2 x_1^\dagger x_2$$
$$\Rightarrow \lambda_1 \neq \lambda_2 \Rightarrow x_1^\dagger x_2 = 0$$

□

Hermitian matrices are diagonizable

Lemma

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Proof.

We will sketch the proof by construction. For any n-d Hermitian matrix M, consider an eigenvalue λ and corresponding eigenvector u, without loss of generality, let's also normalize u such that $\|u\| = 1$. Consider the subspace orthogonal to u, $U^\perp$, and let $v_1, \cdots, v_{n-1}$ be arbitrary orthonormal basis of $U^\perp$. Note that for any k, Av_k will be orthogonal to u since

$$u^\dagger M v_k = u^\dagger M^\dagger v_k = (Mu)^\dagger v_k = \lambda u^\dagger v_k = 0.$$

Thus, $(u, v_1, \cdots, v_{n-1})^\dagger M (u, v_1, \cdots, v_{n-1}) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & M' \end{smallmatrix}\right)$. Moreover, M' is also a Hermitian matrix with one less dimension. We can apply the same process on M' and "diagonalize" one more row/column. That is, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & P' \end{smallmatrix}\right)^\dagger P^\dagger M P \left(\begin{smallmatrix} 1 & 0 \\ 0 & P' \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda & 0 & \cdots \\ 0 & \lambda' & \\ & & M'' \end{smallmatrix}\right)$. We can repeat this until the entire M is diagonalized □

Hermitian matrices are diagonalizable

Remark

A Hermitian matrix is diagonalized by its eigenvectors and the diagonalized matrix is composed of the corresponding eigenvalues. That is,

$$(v_1, \cdots, v_n)^\dagger M \underbrace{(v_1, \cdots, v_n)}_{V} = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}.$$

Moreover, V is unitary (orthogonal), i.e., $V^\dagger V = I$ and thus $V^{-1} = V^\dagger$

Remark

Recall that real-symmetric matrices are Hermitian, thus can be diagonalized by its eigenvectors also

Positive definite matrices

Definition (Positive definite)

For a Hermitian matrix M, it is positive definite iff $\forall x,\ x^\dagger Mx > 0$

Definition (Positive semi-definite)

For a Hermitian matrix M, it is positive semi-definite iff $\forall x,\ x^\dagger Mx \geq 0$

Remark

M is positive definite (semi-definite) iff all its eigenvalue is larger (larger or equal to) 0

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Proof.

$\Rightarrow$: assume positive definite but some eigenvalue < 0, WLOG, let $\lambda_1 < 0$, then $v_1^\dagger M v_1 = \lambda_1 < 0$ contradicts that M is positive definite

$\Leftarrow$: If $\forall k, \lambda_k > 0$, for any x,

$$x^\dagger Mx = (V^\dagger x)^\dagger \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{pmatrix} V^\dagger x = \sum_i \lambda_i (V^\dagger x)_i^2 > 0$$

□

Some probability basic

- Probability mass function (pmf) for discrete random variable (r.v.) X
  - $p(x) \geq 0$
  - $p(x) \leq 1$
  - $\sum_x p(x) = 1$

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- Markov property and conditional independence: $p(x, y|z) = p(x|z)p(y|z)$, $X \perp\!\!\!\perp Y|Z, X \leftrightarrow Z \leftrightarrow Y$

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- Inference: ML, MAP, Bayesian

Inference

o: (Observed) evidence, θ: Parameter, x: prediction

Maximum Likelihood (ML)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(o|\theta)$

Maximum A Posteriori (MAP)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(\theta|o)$

Bayesian

$$\hat{x} = \sum_x x \underbrace{\sum_\theta p(x|\theta)p(\theta|o)}_{p(x|o)}$$

where $p(\theta|o) = \frac{p(o|\theta)p(\theta)}{p(o)} \propto p(o|\theta)\underbrace{p(\theta)}_{prior}$

Covariance matrices

Definition (Covariance matrices)

Recall that for a vector random variable $\boldsymbol{X} = [X_1, X_2, \cdots, X_n]^T$, the covariance matrix $\Sigma \triangleq E[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T]$

Remark

Covariance matrices are always positive semi-definite since $\forall u$,
$u^T \Sigma u = E[u^T(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T u] = E[\|(\boldsymbol{X}-\boldsymbol{\mu})^T u\|^2] \geq 0$

Remark

In general, we usually would like to assume Σ to be strictly positive definite. Because otherwise it means that some of its eigenvalues are zero and so in some dimension, there is actually no variation and is just constant along that dimension. Representing those dimension as random variable is troublesome since "$1/\sigma^2$" which occurs often will become infinite. Instead we can always simply strip away those dimensions to avoid complications

Marginalization of normal distribution

- Consider $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_\mathbf{Z}, \Sigma_\mathbf{Z})$ and let say $\mathbf{X}$ is a segment of $\mathbf{Z}$. That is, $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$ for some $\mathbf{Y}$. Then how should $\mathbf{X}$ behave?

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- We can find the pdf of $\mathbf{X}$ by just marginalizing that of $\mathbf{Z}$. That is

$$
\begin{aligned}
p(\mathbf{x}) &= \int p(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
&= \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int \exp\left(-\frac{1}{2} \begin{pmatrix} \mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} \mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix} \right) d\mathbf{y}
\end{aligned}
$$

Marginalization of normal distribution

- Denote Σ^{-1} as Λ (also known as the precision matrix). And partition both Σ and Λ into $\Sigma = \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \Lambda_{\mathbf{XX}} & \Lambda_{\mathbf{XY}} \\ \Lambda_{\mathbf{YX}} & \Lambda_{\mathbf{YY}} \end{pmatrix}$

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- Then we have

$$
\begin{aligned}
p(\mathbf{x}) = & \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int \exp\left(-\frac{1}{2}\left[(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XX}} (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}) \right.\right. \\
& + (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YX}} (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}) + (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XY}} (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}) \\
& \left.\left. + (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YY}} (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})\right]\right) d\mathbf{y} \\
= & \frac{e^{-\frac{(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XX}} (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int \exp\left(-\frac{1}{2}\left[(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YX}} (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}) \right.\right. \\
& \left.\left. + (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XY}} (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}) + (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YY}} (\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})\right]\right) d\mathbf{y}
\end{aligned}
$$

Marginalization of normal distribution

To proceed, let's apply the completing square trick on $(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YX}}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})+(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T\Lambda_{\mathbf{XY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})+(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})$. For the ease of exposition, let us denote $\tilde{\mathbf{x}}$ as $\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}$ and $\tilde{\mathbf{y}}$ as $\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}$. We have

Marginalization of normal distribution

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$$\begin{aligned}&\tilde{\mathbf{y}}^T\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}}+\tilde{\mathbf{x}}^T\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}}+\tilde{\mathbf{y}}^T\Lambda_{\mathbf{YY}}\tilde{\mathbf{y}}\\ =&(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})-\tilde{\mathbf{x}}^T\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}},\end{aligned}$$

where we use the fact that $\Lambda=\Sigma^{-1}$ is symmetric and so $\Lambda_{\mathbf{XY}}=\Lambda_{\mathbf{YX}}$

Marginalization of normal distribution

$$p(\mathbf{x}) = \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}} - \Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}} d\mathbf{y}$$

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$$= \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}} - \Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}\right)$$

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$$\stackrel{(a)}{=} \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right)$$

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$$\stackrel{(b)}{=} \frac{1}{\sqrt{\det(2\pi\Sigma_{\mathbf{XX}})}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right)$$

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$$= \frac{1}{\sqrt{\det(2\pi\Sigma_{\mathbf{XX}})}} \exp\left(-\frac{(\mathbf{x}-\mu_{\mathbf{X}})^T\Sigma_{\mathbf{XX}}^{-1}(\mathbf{x}-\mu_{\mathbf{X}})}{2}\right),$$

where (a) and (b) will be shown next

(a) $\Sigma_{XX}^{-1} = \Lambda_{XX} - \Lambda_{XY}\Lambda_{YY}^{-1}\Lambda_{YX}$

Proof.

Since $\Lambda = \Sigma^{-1}$, we have $\Sigma_{XX}\Lambda_{XY} + \Sigma_{XY}\Lambda_{YY} = 0$ and $\Sigma_{XX}\Lambda_{XX} + \Sigma_{XY}\Lambda_{YX} = I$. Insert an identity into the latter equation, we have $\Sigma_{XX}\Lambda_{XX} + \Sigma_{XY}(\Lambda_{YY}\Lambda_{YY}^{-1})\Lambda_{YX} = \Sigma_{XX}\Lambda_{XX} - (\Sigma_{XX}\Lambda_{XY})\Lambda_{YY}^{-1}\Lambda_{YX} = \Sigma_{XX}(\Lambda_{XX} - \Lambda_{XY}\Lambda_{YY}^{-1}\Lambda_{YX}) = I$. □

Remark

By symmetry, we also have

$$\Lambda_{XX}^{-1} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

(b') $\det(\Sigma) = \det(\Sigma_{\mathbf{YY}})\det(\Lambda_{\mathbf{XX}}^{-1})$

Proof.

$$\det(\Sigma) = \det\begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$$

□

(b') $\det(\Sigma) = \det(\Sigma_{\mathbf{YY}})\det(\Lambda_{\mathbf{XX}}^{-1})$

Proof.

$$\det(\Sigma) = \det\begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix}\begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix}\right)$$

□

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$$\det(\Sigma) = \det \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$$

$$= \det \left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \right)$$

$$= \det \left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \right)$$

□

(b') $\det(\Sigma) = \det(\Sigma_{\mathbf{YY}}) \det(\Lambda_{\mathbf{XX}}^{-1})$

Proof.

$$\begin{aligned}
\det(\Sigma) &= \det \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix} \\
&= \det \left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \right) \\
&= \det \left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \right) \\
&= \det \begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \det \begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I \end{pmatrix} \det \begin{pmatrix} \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix}
\end{aligned}$$

□

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Proof.

$$
\begin{aligned}
\det(\Sigma) &= \det \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix} \\
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&= \det \left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \right) \\
&= \det \begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}} \end{pmatrix} \det \begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I \end{pmatrix} \det \begin{pmatrix} \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I \end{pmatrix} \\
&= \det \Sigma_{\mathbf{YY}} \det(\Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})
\end{aligned}
$$

□

(b') $\det(\Sigma) = \det(\Sigma_{\mathbf{YY}})\det(\Lambda^{-1}_{\mathbf{XX}})$

Proof.

$$\begin{aligned}
\det(\Sigma) &= \det\begin{pmatrix}\Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}}\end{pmatrix} \\
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&= \det\left(\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}}\end{pmatrix}\begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I\end{pmatrix}\begin{pmatrix}\Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I\end{pmatrix}\right) \\
&= \det\begin{pmatrix} I & 0 \\ 0 & \Sigma_{\mathbf{YY}}\end{pmatrix}\det\begin{pmatrix} I & \Sigma_{\mathbf{XY}} \\ 0 & I\end{pmatrix}\det\begin{pmatrix}\Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & 0 \\ \Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} & I\end{pmatrix} \\
&= \det\Sigma_{\mathbf{YY}}\det(\Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}}) \\
&= \det\Sigma_{\mathbf{YY}}\det\Lambda_{\mathbf{XX}}^{-1},
\end{aligned}$$

where the last equality is from (a) □

(b) $\det(a\Sigma) = \det(a\Sigma_{\mathbf{YY}})\det(a\Lambda_{\mathbf{XX}}^{-1})$ for any constant a

Proof.

Note that since the width (height) of Σ is equal to the sum of the widths of $\Sigma_{\mathbf{XX}}$ and $\Sigma_{\mathbf{YY}}$. The equation below follows immediately □

Remark

Note that by symmetry, we also have $\det(a\Sigma) = \det(a\Sigma_{\mathbf{XX}})\det(a\Lambda_{\mathbf{YY}}^{-1})$ *for any constant a. Take* $a = 2\pi$ *and that is exactly what we need for (b)*

Conditioning of normal distribution

- Consider the same $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$. What will $\mathbf{X}$ be like if $\mathbf{Y}$ is observed to be $\mathbf{y}$?

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- Basically, we want to find $p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}, \mathbf{y})/p(\mathbf{y})$

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- Basically, we want to find $p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}, \mathbf{y})/p(\mathbf{y})$
- From previous result, we have $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}; \boldsymbol{\mu}_\mathbf{Y}, \Sigma_{\mathbf{YY}})$. Therefore,

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}\left[\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{pmatrix} - \tilde{\mathbf{y}}^T \Sigma_{\mathbf{YY}}^{-1} \tilde{\mathbf{y}}\right]\right)$$
$$\propto \exp\left(-\frac{1}{2}[\tilde{\mathbf{x}}^T \Lambda_{\mathbf{XX}} \tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T \Lambda_{\mathbf{XY}} \tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T \Lambda_{\mathbf{YX}} \tilde{\mathbf{x}}]\right),$$

where we use $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ as shorthands of $\mathbf{x} - \boldsymbol{\mu}_\mathbf{X}$ and $\mathbf{y} - \boldsymbol{\mu}_\mathbf{Y}$ as before

Conditioning of normal distribution

- Completing the square for $\tilde{\mathbf{x}}$, we have

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})^T\Lambda_{\mathbf{XX}}(\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})\right)$$

$$= \exp\Big(-\frac{1}{2}(\mathbf{x} - \mu_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \mu_{\mathbf{Y}}))^T\Lambda_{\mathbf{XX}}$$

$$(\mathbf{x} - \mu_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \mu_{\mathbf{Y}}))\Big)$$

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$$= \exp\left(-\frac{1}{2}(\mathbf{x} - \mu_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \mu_{\mathbf{Y}}))^T\Lambda_{\mathbf{XX}} (\mathbf{x} - \mu_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \mu_{\mathbf{Y}}))\right)$$

- Therefore $\mathbf{X}|\mathbf{y}$ is Gaussian distributed with mean $\mu_{\mathbf{X}} - \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \mu_{\mathbf{Y}})$ and covariance $\Lambda_{\mathbf{XX}}^{-1}$

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$$= \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))^T \Lambda_{\mathbf{XX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))\right)$$

- Therefore $\mathbf{X}|\mathbf{y}$ is Gaussian distributed with mean $\boldsymbol{\mu}_{\mathbf{X}} - \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})$ and covariance $\Lambda_{\mathbf{XX}}^{-1}$
- Note that since $\Lambda_{\mathbf{XX}}\Sigma_{\mathbf{XY}} + \Lambda_{\mathbf{XY}}\Sigma_{\mathbf{YY}} = 0$, $\Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}} = -\Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}$ and from (a), we have

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

Interpretation of conditioning

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\mu_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \mu_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

- When the observation of **Y** is exactly the mean, the conditioned mean does not change

Interpretation of conditioning

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- When the observation of **Y** is exactly the mean, the conditioned mean does not change
- Otherwise, it needs to be modified and the size of the adjustment decreases with $\Sigma_{\mathbf{YY}}$, the variance of **Y** for the 1-D case.
  - The observation is less reliable with the increase of $\Sigma_{\mathbf{YY}}$. The adjustment is finally scaled by $\Sigma_{\mathbf{XY}}$, which translates the variation of **Y** to the variation of **X**

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- When the observation of $\mathbf{Y}$ is exactly the mean, the conditioned mean does not change
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    - In particular, if $\mathbf{X}$ and $\mathbf{Y}$ are negatively correlated, the sign of the adjustment will be reversed

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- When the observation of **Y** is exactly the mean, the conditioned mean does not change
- Otherwise, it needs to be modified and the size of the adjustment decreases with $\Sigma_{\mathbf{YY}}$, the variance of **Y** for the 1-D case.
  - The observation is less reliable with the increase of $\Sigma_{\mathbf{YY}}$. The adjustment is finally scaled by $\Sigma_{\mathbf{XY}}$, which translates the variation of **Y** to the variation of **X**
  - In particular, if **X** and **Y** are negatively correlated, the sign of the adjustment will be reversed
- As for the variance of the conditioned variable, it always decreases and the decrease is larger if $\Sigma_{\mathbf{YY}}$ is smaller and $\Sigma_{\mathbf{XY}}$ is larger (**X** and **Y** are more correlated)

$X \perp\!\!\!\perp Y|Z$ if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$

Corollary

Given multivariate Gaussian variables X, Y and Z, we have X and Y are conditionally independent given Z if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$, where $\rho_{XZ} = \frac{E[(X-E(X))(Z-E(Z))]}{\sqrt{E[(X-E(X))^2]E[(Z-E(Z))^2]}}$ is the correlation coefficent between X and Z. Similarly, ρ_{YZ} and ρ_{XY} are the correlation coefficients between Y and Z, and X and Y, respectively.

$X \perp\!\!\!\perp Y | Z$ if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$

Proof.

- Without loss of generality, we can assume the variables with mean 0 and variance 1. Thus, $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma)$, where $\Sigma = \begin{pmatrix} 1 & \rho_{XY} & \rho_{XZ} \\ \rho_{XY} & 1 & \rho_{YZ} \\ \rho_{XZ} & \rho_{YZ} & 1 \end{pmatrix}$

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- Then from the conditioning result, we have

$$\Sigma_{\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)|Z} = \begin{pmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{pmatrix} - \begin{pmatrix} \rho_{XZ} & \rho_{YZ} \end{pmatrix} \sigma_{YY}^{-1} \begin{pmatrix} \rho_{XZ} \\ \rho_{YZ} \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \rho_{XZ}^2 & \rho_{XY} - \rho_{XZ}\rho_{YZ} \\ \rho_{XY} - \rho_{XZ}\rho_{YZ} & 1 - \rho_{YZ}^2 \end{pmatrix}$$

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Proof.

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$$= \begin{pmatrix} 1 - \rho_{XZ}^2 & \rho_{XY} - \rho_{XZ}\rho_{YZ} \\ \rho_{XY} - \rho_{XZ}\rho_{YZ} & 1 - \rho_{YZ}^2 \end{pmatrix}$$

- Therefore, X and Y are uncorrelated given Z when $\sigma_{XY|Z} = \rho_{XY} - \rho_{XZ}\rho_{YZ} = 0$ or $\rho_{XY} = \rho_{XZ}\rho_{YZ}$. Since for Gaussian variables, uncorrelatedness implies independence. This concludes the proof.

□

Product of normal distributions

- Assume that we tries to recover some vector parameter $\mathbf{x}$, which is subject to multivariate Gaussian noise

Product of normal distributions

- Assume that we tries to recover some vector parameter $\mathbf{x}$, which is subject to multivariate Gaussian noise
- Say we made two measurements $\mathbf{y}_1$ and $\mathbf{y}_2$, where $\mathbf{Y}_1 \sim \mathcal{N}(\mathbf{x}, \Sigma_{\mathbf{Y}_1})$ and $\mathbf{Y}_2 \sim \mathcal{N}(\mathbf{x}, \Sigma_{\mathbf{Y}_2})$. Note that even though both measurements have mean $\mathbf{x}$, they have different covariance
  - This variation, for instance, can be due to environment change between the two measurements

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  - This variation, for instance, can be due to environment change between the two measurements
- Now, if we want to compute the overall likelihood, $p(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x})$. Assuming that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are conditionally independent given $\mathbf{X}$, we have

$$\begin{aligned} p(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}) &= p(\mathbf{y}_1|\mathbf{x})p(\mathbf{y}_2|\mathbf{x}) \\ &= \mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}). \end{aligned}$$

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- Essentially, we just need to compute the product of two Gaussian pdfs. Such computation is very useful and it occurs often when one needs to perform inference

Product of normal distributions

As in previous cases, the product turns out to be normal also. However, unlike them, the product is not a pdf and so it does not normalize to 1. So we have to compute both the scaling factor and the exponent explicitly. Let us start with the exponent.

$$\mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2})$$

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$$\propto \exp\left(-\frac{1}{2}[(\mathbf{x}-\mathbf{y}_1)^T\Lambda_{\mathbf{Y}_1}(\mathbf{x}-\mathbf{y}_1)+(\mathbf{x}-\mathbf{y}_2)^T\Lambda_{\mathbf{Y}_2}(\mathbf{x}-\mathbf{y}_2)]\right)$$

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\begin{aligned}
&\mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}) \\
&\propto \exp\left(-\frac{1}{2}[(\mathbf{x}-\mathbf{y}_1)^T\Lambda_{\mathbf{Y}_1}(\mathbf{x}-\mathbf{y}_1) + (\mathbf{x}-\mathbf{y}_2)^T\Lambda_{\mathbf{Y}_2}(\mathbf{x}-\mathbf{y}_2)]\right) \\
&\propto \exp\left(-\frac{1}{2}[\mathbf{x}^T(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})\mathbf{x} - (\mathbf{y}_2^T\Lambda_{\mathbf{Y}_2} + \mathbf{y}_1^T\Lambda_{\mathbf{Y}_1})\mathbf{x} - \mathbf{x}^T(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2 + \Lambda_{\mathbf{Y}_1}\mathbf{y}_1)]\right)
\end{aligned}
$$

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&\propto \exp\left(-\frac{1}{2}[\mathbf{x}^T(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})\mathbf{x}-(\mathbf{y}_2^T\Lambda_{\mathbf{Y}_2}+\mathbf{y}_1^T\Lambda_{\mathbf{Y}_1})\mathbf{x}-\mathbf{x}^T(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1)]\right) \\
&\propto e^{-\frac{1}{2}[(\mathbf{x}-(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1))^T(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})(\mathbf{x}-(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1))]}
\end{aligned}
$$

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&\mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}) \\
\propto &\exp\left(-\frac{1}{2}[(\mathbf{x}-\mathbf{y}_1)^T\Lambda_{\mathbf{Y}_1}(\mathbf{x}-\mathbf{y}_1) + (\mathbf{x}-\mathbf{y}_2)^T\Lambda_{\mathbf{Y}_2}(\mathbf{x}-\mathbf{y}_2)]\right) \\
\propto &\exp\left(-\frac{1}{2}[\mathbf{x}^T(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})\mathbf{x} - (\mathbf{y}_2^T\Lambda_{\mathbf{Y}_2}+\mathbf{y}_1^T\Lambda_{\mathbf{Y}_1})\mathbf{x} - \mathbf{x}^T(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1)]\right) \\
\propto &e^{-\frac{1}{2}[(\mathbf{x}-(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1))^T(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})(\mathbf{x}-(\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1))]} \\
\propto &\mathcal{N}(\mathbf{x}; (\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1), (\Lambda_{\mathbf{Y}_2}+\Lambda_{\mathbf{Y}_1})^{-1})
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}) \\
= &K(\mathbf{y}_1, \mathbf{y}_2, \Sigma_{\mathbf{Y}_1}, \Sigma_{\mathbf{Y}_2})\mathcal{N}(\mathbf{x}; (\Lambda_{\mathbf{Y}_1}+\Lambda_{\mathbf{Y}_2})^{-1}(\Lambda_{\mathbf{Y}_2}\mathbf{y}_2+\Lambda_{\mathbf{Y}_1}\mathbf{y}_1), (\Lambda_{\mathbf{Y}_2}+\Lambda_{\mathbf{Y}_1})^{-1})
\end{aligned}$$

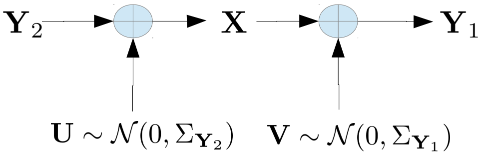
for some scaling factor $K(\mathbf{y}_1, \mathbf{y}_2, \Sigma_{\mathbf{Y}_1}, \Sigma_{\mathbf{Y}_2})$ independent of $\mathbf{x}$.

Product of normal distributions

- One can compute the scaling factor $K(\mathbf{y}_1, \mathbf{y}_2, \Sigma_{\mathbf{Y}_1}, \Sigma_{\mathbf{Y}_2})$ directly

Product of normal distributions

- One can compute the scaling factor $K(\mathbf{y}_1, \mathbf{y}_2, \Sigma_{\mathbf{Y}_1}, \Sigma_{\mathbf{Y}_2})$ directly
- However, it is much easier to take advantage for the following setup when $\mathbf{X} \perp\!\!\!\perp \mathbf{Y_1} | \mathbf{Y_2}$ as shown below



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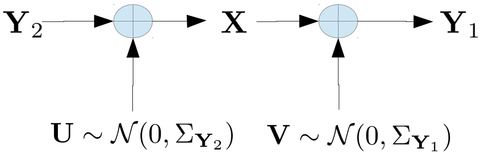

- Since $\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}) = \mathcal{N}(\mathbf{x}; \mathbf{y}_2, \Sigma_{\mathbf{Y}_2})$ and $\mathbf{X} \perp\!\!\!\perp \mathbf{Y_1} | \mathbf{Y_2}$, we have

$$\mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{y}_2; \mathbf{x}, \Sigma_{\mathbf{Y}_2}) = \mathcal{N}(\mathbf{y}_1; \mathbf{x}, \Sigma_{\mathbf{Y}_1})\mathcal{N}(\mathbf{x}; \mathbf{y}_2, \Sigma_{\mathbf{Y}_2}) = p(\mathbf{y}_1, \mathbf{x} | \mathbf{y}_2)$$

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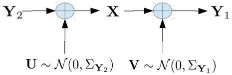
- Then, marginalizing $\mathbf{x}$ out from $p(\mathbf{y}_1, \mathbf{x}|\mathbf{y}_2)$, we have $p(\mathbf{y}_1|\mathbf{y}_2) = \int p(\mathbf{y}_1, \mathbf{x}|\mathbf{y}_2)d\mathbf{x}$. However, from the figure,



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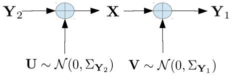

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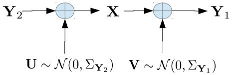

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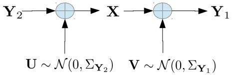

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- The scaling factor, $\mathcal{N}(\mathbf{y}_1; \mathbf{y}_2, \Sigma_{\mathbf{Y}_2} + \Sigma_{\mathbf{Y}_1})$, can be interpreted as how much one can believe on the overall likelihood.
  - The value is reasonable since when the two observations are far away with respect to the overall variance $\Sigma_{\mathbf{Y}_2} + \Sigma_{\mathbf{Y}_1}$, the likelihood will become less reliable
  - The scaling factor is especially useful when we deal with mixture of Gaussian to be discussed next

Division of normal distributions

- To compute $\frac{\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_1,\Sigma_1)}{\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_2,\Sigma_2)}$, note that from the product formula earlier

$$\begin{aligned}&\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_2,\Sigma_2)\mathcal{N}(\mathbf{x};(\Lambda_1-\Lambda_2)^{-1}(\Lambda_1\boldsymbol{\mu}_1-\Lambda_2\boldsymbol{\mu}_2),(\Lambda_1-\Lambda_2)^{-1})\\ =&\mathcal{N}(\boldsymbol{\mu}_1;(\Lambda_1-\Lambda_2)^{-1}(\Lambda_1\boldsymbol{\mu}_1-\Lambda_2\boldsymbol{\mu}_2),\Lambda_2^{-1}+(\Lambda_1-\Lambda_2)^{-1})\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_1,\Sigma_1)\end{aligned}$$

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- Therefore,

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  where $\boldsymbol{\mu}=(\Lambda_1-\Lambda_2)^{-1}(\Lambda_1\boldsymbol{\mu}_1-\Lambda_2\boldsymbol{\mu}_2)$

- Note that the final pdf will be Gaussian-like if $\Lambda_1 \succeq \Lambda_2$. Otherwise, one can still write out the pdf using the precision matrix. But the covariance matrix will not be defined (Try plot some pdfs out yourselves)

Mixture of Gaussians

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- When the system is on, the output signal S behaves like $\mathcal{N}(5,1)$. When the system is off is off, S behaves like $\mathcal{N}(0,1)$

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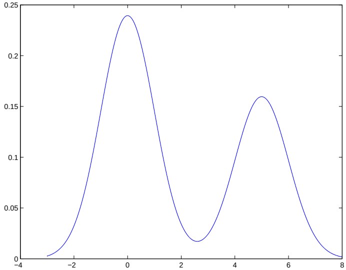
- The pdf of S will be $0.4\mathcal{N}(s;5,1)+0.6\mathcal{N}(s;0,1)$ as shown below



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- Let us illustrate this with the following example:
  - Consider two mixtures of Gaussian likelihood of x given two observations y_1 and y_2 as follows:

$$p(y_1|x) = 0.6\mathcal{N}(x; 0, 1) + 0.4\mathcal{N}(x; 5, 1);$$
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- As usual, it is reasonable to assume the observations to be conditionally independent given x. Then,

$$\begin{aligned} p(y_1, y_2|x) &= p(y_1|x)p(y_2|x) \\ &= (0.6\mathcal{N}(x; 0, 1) + 0.4\mathcal{N}(x; 5, 1))(0.5\mathcal{N}(x; -2, 1) + 0.5\mathcal{N}(x; 4, 1)) \\ &= 0.3\mathcal{N}(x; 0, 1)\mathcal{N}(x; -2, 1) + 0.2\mathcal{N}(x; 5, 1)\mathcal{N}(x; -2, 1) \\ &\quad + 0.3\mathcal{N}(x; 0, 1)\mathcal{N}(x; 4, 1) + 0.2\mathcal{N}(x; 5, 1)\mathcal{N}(x; 4, 1) \end{aligned}$$

Explosion of Gaussians

- The last step involves computing products of Gaussians but we have learned it in previous sections. Using the previous result,

$$\begin{aligned} p(y_1, y_2|x) = &\ 0.3\mathcal{N}(-2; 0, 2)\mathcal{N}(x; -1, 0.5) + 0.2\mathcal{N}(-2; 5, 2)\mathcal{N}(x; 1.5, 0.5) \\ &+ 0.3\mathcal{N}(4; 0, 2)\mathcal{N}(x; 2, 0.5) + 0.2\mathcal{N}(4; 5, 2)\mathcal{N}(x; 4.5, 0.5). \end{aligned}$$

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- Let's repeat our discussion but with n observations instead. The overall likelihood will be a mixture of 2^n Gaussians!
  - Therefore, the computation will quickly become intractable as the number of observations increases
  - Fortunately, in reality, some of the Gaussians in the mixture tend to have a very small weight

Reduce number of components in Gaussian mixtures

- For instance, in our previous numerical example, if we continue our numerical computation for the two observation example, we have

$$
\begin{aligned}
p(y_1, y_2|x) = 0.4163\mathcal{N}(x; -1, 0.5) + 3.5234 \times 10^{-6}\mathcal{N}(x; 1.5, 0.5) \\
+ 0.0202\mathcal{N}(x; 2, 0.5) + 0.5734\mathcal{N}(x; 4.5, 0.5).
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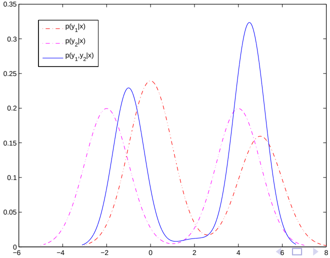
- We can see that the weight for the component at mean 1.5 is very small. And the component at mean 2 has a rather small weight also.
- Even with the four Gaussian components, the overall likelihood is essentially just a bimodal distribution as shown in the figure below



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- Therefore, we may approximate $p(y_1, y_2|x)$ with only two of its original component as $0.4163/(0.4163 + 0.5734)\mathcal{N}(x; -1, 0.5) + 0.5734/(0.4163 + 0.5734)\mathcal{N}(x; 4.5, 0.5) = 0.4206\mathcal{N}(x; -1, 0.5) + 0.5794\mathcal{N}(x; 4.5, 0.5)$

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- However, it is not always a good approximation strategy just to dump away the small components in a Gaussian mixture

Another example

Consider

$$p(x) = 0.1\mathcal{N}(x; -0.2, 1) + 0.1\mathcal{N}(x; -0.1, 1) + 0.1\mathcal{N}(x; 0, 1) + 0.1\mathcal{N}(x; 0.1, 1) + 0.1\mathcal{N}(x; 0.2, 1) + 0.5\mathcal{N}(x; 5, 1).$$

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- Let say we want to reduce $p(x)$ to only a mixture of two Gaussians. It is tempting to just dumping four smallest one and renormalized the weight. For example, if we choose to remove the first four components, we have

$$\hat{p}(x) = 1/6\mathcal{N}(x; 0.2, 1) + 5/6\mathcal{N}(x; 5, 1)$$

Another example

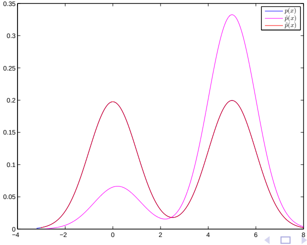
Consider

$$p(x) = 0.1\mathcal{N}(x; -0.2, 1) + 0.1\mathcal{N}(x; -0.1, 1) + 0.1\mathcal{N}(x; 0, 1) + 0.1\mathcal{N}(x; 0.1, 1) + 0.1\mathcal{N}(x; 0.2, 1) + 0.5\mathcal{N}(x; 5, 1).$$

- Let say we want to reduce $p(x)$ to only a mixture of two Gaussians. It is tempting to just dumping four smallest one and renormalized the weight. For example, if we choose to remove the first four components, we have

$$\hat{p}(x) = 1/6\mathcal{N}(x; 0.2, 1) + 5/6\mathcal{N}(x; 5, 1)$$

- The approximation $\hat{p}(x)$ is significantly different from $p(x)$ as shown below



Merging components

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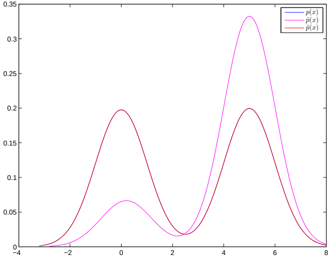
- Actually the first five components are so similar that their combined contribution can be accurately modeled as one Gaussian
- So rather than discarding the components, one can get a much more accurate approximation by merging them. The approximation is illustrated as $\tilde{p}(x)$ in the figure below



Merging components

To successfully obtain such approximation $\tilde{p}(x)$, we have to answer two questions:

- which components to merge?
- how to merge them?

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- The inner product maximizes $(= 1)$ when $p(\mathbf{x}) = q(\mathbf{x})$. This suggests a very reasonable similarity measure between two pdfs

Similarity measure

- Let's define

$$Sim(p(\mathbf{x}), q(\mathbf{x})) \triangleq \frac{\int p(\mathbf{x})q(\mathbf{x})d\mathbf{x}}{\sqrt{\int p(\mathbf{x})^2 d\mathbf{x} \int q(\mathbf{x})^2 d\mathbf{x}}}$$

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- In particular, if $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_p, \Sigma_p)$ and $q(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_q, \Sigma_q)$, we have (please verify)

$$Sim(\mathcal{N}(\boldsymbol{\mu}_p, \Sigma_p), \mathcal{N}(\boldsymbol{\mu}_q, \Sigma_q)) = \frac{\mathcal{N}(\boldsymbol{\mu}_p; \boldsymbol{\mu}_q, \Sigma_p + \Sigma_q)}{\sqrt{\mathcal{N}(0; 0, 2\Sigma_p)\mathcal{N}(0; 0, 2\Sigma_q)}},$$

which can be computed very easily and is equal to one only when means and covariances are the same

How to Merge Components?

Say we have n components $\mathcal{N}(\boldsymbol{\mu}_1, \Sigma_1)$, $\mathcal{N}(\boldsymbol{\mu}_2, \Sigma_2)$, $\cdots$, $\mathcal{N}(\boldsymbol{\mu}_n, \Sigma_n)$ with weights $w_1, w_2, \cdots, w_n$. What should the combined component be like?

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  - Because the weighted sum only counted the contribution of variation among each component, it did not take into account the variation due to different means across components.
  - Instead, let's denote $\mathbf{X}$ as the variable sampled from the mixture. That is, $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}_i, \Sigma_i)$ with probability $\hat{w}_i$. Then, we have (please verify)

$$\begin{aligned} \Sigma &= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ &= \sum_{i=1}^n \hat{w}_i(\Sigma_i + \boldsymbol{\mu}_i\boldsymbol{\mu}_i^T) - \sum_{i=1}^n \sum_{j=1}^n \hat{w}_i \hat{w}_j \boldsymbol{\mu}_i \boldsymbol{\mu}_j^T. \end{aligned}$$

Now, go back to our previous numerical example

- Recall that $p(x) = 0.1\mathcal{N}(x; -0.2, 1) + 0.1\mathcal{N}(x; -0.1, 1) + 0.1\mathcal{N}(x; 0, 1) + 0.1\mathcal{N}(x; 0.1, 1) + 0.1\mathcal{N}(x; 0.2, 1) + 0.5\mathcal{N}(x; 5, 1)$

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- If we merge the five smallest components (one can easily check that they are also more similar to each other than to the last component), we have $\tilde{p}(x) = 0.5\mathcal{N}(x; 0, 1.02) + 0.5\mathcal{N}(x; 5, 1)$ as shown again below. The approximate pdf is virtually indistinguishable from the original