Conjugate prior summary

Distribution	Likelihood $p(\mathbf{x}\|\theta)$	Prior $p(\theta)$	Distribution
Bernoulli	$(1-\theta)^{(1-x)}\theta^x$	$\propto (1-\theta)^{(a-1)}\theta^{(b-1)}$	Beta
Binomial	$\propto (1-\theta)^{(N-x)}\theta^x$	$\propto (1-\theta)^{(a-1)}\theta^{(b-1)}$	Beta
Multinomial	$\propto \theta_1^{x_1}\theta_2^{x_2}\theta_3^{x_3}$	$\propto \theta_1^{\alpha_1-1}\theta_2^{\alpha_2-1}\theta_3^{\alpha_3-1}$	Dirichlet
Normal (fixed σ^2)	$\propto \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)$	$\propto \exp\left(-\frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right)$	Normal
Normal (fixed μ)	$\propto \sqrt{\theta}\exp\left(-\frac{\theta(x-\mu)^2}{2}\right)$	$\propto \theta^{a-1}exp(-b\theta)$	Gamma
Poisson	$\propto \theta^x\exp(-\theta)$	$\propto \theta^{a-1}exp(-b\theta)$	Gamma

An example

- Simple economy: m prosumers, n different goods[1]
- Each individual: production $\mathbf{p}_i \in \mathbb{R}_n$, consumption $\mathbf{c}_i \in \mathbb{R}_n$
- Expense of producing "$\mathbf{p}$" for agent $i = e_i(\mathbf{p})$
- Utility (happiness) of consuming "$\mathbf{c}$" units for agent $i = u_i(\mathbf{c})$
- Maximize happiness

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_i (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \qquad s.t. \qquad \sum_i \mathbf{c}_i = \sum_i \mathbf{p}_i$$

[1] Example borrowed from the first lecture of Prof Gordon's CMU CS 10-725

Walrasian equilibrium

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_i (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \qquad s.t. \qquad \sum_i \mathbf{c}_i = \sum_i \mathbf{p}_i$$

- Idea: introduce price λ_j to each good j. Let the market decide
  - Price $\lambda_j \uparrow$: consumption of good $j \downarrow$, production of good $j \uparrow$
  - Price $\lambda_j \downarrow$: consumption of good $j \uparrow$, production of good $j \downarrow$
  - Can adjust price until consumption $=$ production for each good

Algorithm: tâtonnement

Assume that the appropriate prices are found, we can ignore the equality constraint, then the problem becomes

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_i (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \quad \Rightarrow \quad \sum_i \max_{\mathbf{p}_i,\mathbf{c}_i} (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i))$$

So we can simply optimize production and consumption of each individual independently

Algorithm 1 tâtonnement

```
procedure FindBestPrices
    λ ← [0, 0, ··· , 0]
    for k = 1, 2, ··· do
        Each individual solves for its c_i and p_i for the given λ
        λ ← λ + δ_k Σ_i (c_i − p_i)
```

Lagrange multiplier

Problem

$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$g(\mathbf{x}) = 0$$

Consider $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$ and let $\tilde{f}(\mathbf{x}) = \min_\lambda L(\mathbf{x}, \lambda)$.

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$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) \text{ if } g(\mathbf{x}) = 0 \\ -\infty \text{ otherwise} \end{cases}$$

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Therefore, the problem is identical to $\max_{\mathbf{x}} \tilde{f}(\mathbf{x})$ or

$$\max_{\mathbf{x}} \min_{\lambda} (f(\mathbf{x}) - \lambda g(\mathbf{x})),$$

where λ is known to be the Lagrange multiplier.

Lagrange multiplier (con't)

Assume the optimum is a saddle point,

$$\max_{\mathbf{x}} \min_{\lambda} (f(\mathbf{x}) - \lambda g(\mathbf{x})) = \min_{\lambda} \max_{\mathbf{x}} (f(\mathbf{x}) - \lambda g(\mathbf{x})),$$

the R.H.S. implies

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$$

Inequality constraint

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Consider $\tilde{f}(\mathbf{x}) = \min_{\lambda \geq 0}(f(\mathbf{x}) - \lambda g(\mathbf{x}))$,

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Therefore, we can rewrite the problem as

$$\max_{\mathbf{x}} \min_{\lambda \geq 0} (f(\mathbf{x}) - \lambda g(\mathbf{x}))$$

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Moreover, at the optimum point $(\mathbf{x}^*, \lambda^*)$, we should have the so-called "complementary slackness" condition

$$\lambda^* g(\mathbf{x}^*) = 0$$

since

$$\max_{\substack{\mathbf{x} \\ g(\mathbf{x}) \leq 0}} f(\mathbf{x}) \equiv \max_{\mathbf{x}} \min_{\lambda \geq 0} (f(\mathbf{x}) - \lambda g(\mathbf{x}))$$

Karush-Kuhn-Tucker conditions

Problem

$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$g(\mathbf{x}) \leq 0, \quad h(\mathbf{x}) = 0$$

Conditions

$$\begin{aligned} \nabla f(\mathbf{x}^*) - \mu^* \nabla g(\mathbf{x}^*) - \lambda^* \nabla h(\mathbf{x}^*) &= 0 \\ g(\mathbf{x}^*) &\leq 0 \\ h(\mathbf{x}^*) &= 0 \\ \mu^* &\geq 0 \\ \mu^* g(\mathbf{x}^*) &= 0 \end{aligned}$$

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- However, we want to make sure that we can losslessly decode the message also!

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  - We say $c(\mathbf{x})$ is uniquely decodable if all input sequences map to different outputs

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    - $a \mapsto 10, b \mapsto 00, c \mapsto 11, d \mapsto 110$

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- Instead, for a mapping $a \mapsto 1, b \mapsto 01, c \mapsto 001, d \mapsto 0001$, I will argue that we can always decode a symbol "once it is available"
  - Note that the catch is that there is no codeword being the "prefix" of another codeword
  - We call such code a prefix-free code or an instantaneous code

Kraft's Inequality

Let $l_1, l_2, \cdots, l_K$ satisfy $\sum_{k=1}^{K} 2^{-l_k} \leq 1$. Then, there exists a uniquely decodable code for symbols $x_1, x_2, \cdots, x_K$ such that $l(x_1) = l_1$, $l(x_2) = l_2, \cdots, l(x_K) = l_K$.

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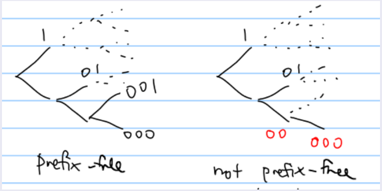
Intuition

Consider # "descendants" of each codeword at the "l_{max}"-level, then for prefix-free code, we have

$$\sum_{k=1}^{K} 2^{l_{max}-l} \leq 2^{l_{max}}$$



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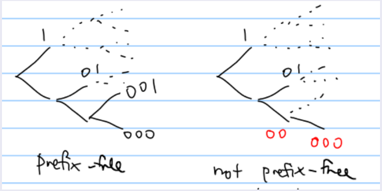

Forward Proof

Given $l_1, l_2, \cdots, l_K$ satisfy $\sum_{k=1}^{K} 2^{-l_k} \leq 1$, we can assign nodes on a tree as previous slides. More precisely,

- Assign i-th node as a node at level l_i, then cross out all its descendants
- Repeat the procedure for i from 1 to K
- We know that there are sufficient tree nodes to be assigned since the Kraft's inequaltiy is satisfied

The corresponding code is apparently prefix-free and thus is uniquely decodable

Converse Proof

Consider message from coding k symbols $\mathbf{x} = x_1, x_2, \cdots, x_k$

$$\left(\sum_{x\in\mathcal{X}} 2^{-l(x)}\right)^k = \left(\sum_{x_1\in\mathcal{X}} 2^{-l(x_1)}\right)\left(\sum_{x_2\in\mathcal{X}} 2^{-l(x_2)}\right)\cdots\left(\sum_{x_k\in\mathcal{X}} 2^{-l(x_k)}\right)$$

$$= \sum_{x_1,x_2,\cdots,x_k\in\mathcal{X}^k} 2^{-(l(x_1)+l(x_2)+\cdots+l(x_k))}$$

$$= \sum_{\mathbf{x}\in\mathcal{X}^k} 2^{-l(\mathbf{x})}$$

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where $a(m)$ is the number of codeword with length m. However, for the code to be uniquely decodable, $a(m) \leq 2^m$, where 2^m is the number of available codewords with length m.

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$$\sum_{x\in\mathcal{X}} 2^{-l(x)} \leq (kl_{max})^{1/k} \approx 1 \text{ as } k \to \infty$$

Minimum rate required to compress a source

$$min_{l_1,l_2,\cdots,l_K} \sum_{k=1}^{K} p_k l_k \text{ subject to } \sum_{k=1}^{K} 2^{-l_k} \leq 1 \text{ and } l_1,\cdots,l_K \geq 0$$

$$\equiv max_{l_1,l_2,\cdots,l_K} -\sum_{k=1}^{K} p_k l_k \text{ subject to } \sum_{k=1}^{K} 2^{-l_k} - 1 \leq 0 \text{ and } -l_1,\cdots,-l_K \leq 0$$

KKT conditions

$$-\nabla\left(\sum_{k=1}^{K} p_k l_k\right) - \mu_0 \nabla\left(\sum_{k=1}^{K} 2^{-l_k} - 1\right) + \sum_{k=1}^{K} \mu_k \nabla l_k = 0$$

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$$\mu_0\left(\sum_{k=1}^{K} 2^{-l_k} - 1\right) = 0, \quad \mu_k l_k = 0$$

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And by $\sum_{k=1}^K 2^{-l_k} \leq 1$, we have

$$\sum_{k=1}^K \frac{p_j}{\mu_0 \log 2} = \frac{1}{\mu_0 \log 2} \leq 1 \Rightarrow \mu_0 \geq \frac{1}{\log 2}$$

Minimum rate required to compress a source

Since we expect $l_k > 0$, $\mu_k = 0$. Expand the first equation, we get

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Note that as $\mu_0 \downarrow$, $\frac{p_j}{\mu_0 \log 2} \uparrow$ and $l_j \downarrow$. Therefore, if we want to decrease code rate, we should reduce μ_0 as much as possible. Thus, take $\mu_0 = \frac{1}{\log 2}$. Then $2^{-l_j} = p_j \Rightarrow l_j = -\log_2 p_j$. Thus, the minimum rate becomes

$$\sum_{k=1}^K p_k l_k = -\sum_{k=1}^K p_k \log_2 p_k \triangleq H(p_1, \cdots, p_K)$$

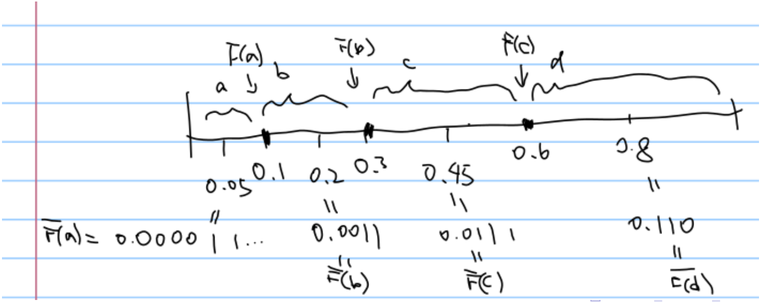
Shannon-Fano-Elias code

Key idea

Each codeword corresponds to an intervel of $[0, 1]$

Example

110 corresponds to $[0.110, 0.110\dot{1}] = [0.11, 0.111) = [0.75, 0.875)$



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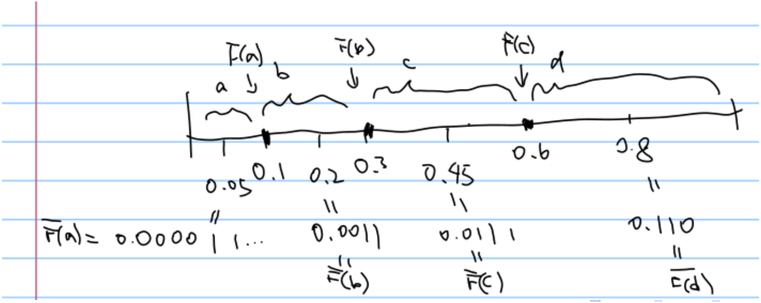

Example

Consider a source that
$p(x_1) = 0.25, p(x_2) = 0.25, p(x_3) = 0.2, p(x_4) = 0.15, p(x_5) = 0.15$

x	$p(x)$	$F(x)$	$\overline{F}(x)$	$\overline{F}(x)$ in Binary	$l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil + 1$	Codeword
1	0.25	0.25	0.125	0.001	3	001
2	0.25	0.5	0.375	0.011	3	011
3	0.2	0.7	0.6	$0.1\overline{0011}$	4	1001
4	0.15	0.85	0.775	$0.110\overline{0011}$	4	1100
5	0.15	1.0	0.925	$0.111\overline{0110}$	4	1110

Property

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  - Since no codeword can overlap in SFE, no code word can be prefix of another
- Average code rate is upper bounded by $H(X) + 2$

$$\sum_{x \in \mathcal{X}} p(x) l(x) = \sum_{x \in \mathcal{X}} p(x) \left(\left\lceil \log_2 \frac{1}{p(x)} \right\rceil + 1 \right)$$
$$\leq \sum_{x \in \mathcal{X}} p(x) \left(\log_2 \frac{1}{p(x)} + 2 \right) = H(X) + 2$$

"Symbol grouping" trick

- Let's consider two symbols as a super-symbol and compress the pair at each time with SFE code
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&= -\sum_{x_1\in\mathcal{X}} p(x_1)\log_2 p(x_1) - \sum_{x_2\in\mathcal{X}} p(x_2)\log_2 p(x_2)\\
&= 2H(X)
\end{aligned}$$

Therefore, the code rate per original symbol is upper bounded by

$$\frac{1}{2}(H(X_S)+2) = H(X)+1$$

Forward proof of Source Coding Theorem

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Therefore as long as a given rate $R > H(X)$, we can always find a large enough N such that the code rate using the "grouping trick" and SFE code is below R. This concludes the forward proof