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- Source coding theorem: For an independent and identically distributed (i.i.d.) discrete memoryless source (DMS) X, we can always compress it with no less than $H(X)$ bits per input symbol, where $H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) = E[-\log p(X)]$

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- For a quantized version of continuous X, $H(X_\Delta) = h(X) - \log\Delta$
- For multivariate normal $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$,

$$h(\boldsymbol{X}) = \log\sqrt{\det(2\pi e\Sigma)}$$

Upper bound of differential entropy

$$h(X) \leq \log E\left[\frac{1}{p(X)}\right] = \log \int_{x\in\mathcal{X}} p(x)\frac{1}{p(x)}dx = \log|\mathcal{X}|$$

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- The expression still makes sense but it is not useful usually since the sampling space can be unbounded $|\mathcal{X}| = \infty$ (for example, normally distributed X)
- Thus it makes much more sense to consider upper bound of a differential entropy constrained on the variance of the variable (why not constrained on mean?)
- It turns out that for a fixed variance σ^2, the variable will have largest differential entropy if it is normally distributed (will show later). Thus

$$h(X) \leq \log\sqrt{2\pi e\sigma^2}$$

Joint entropy

For multivariate random variable, we can extend the definition of entropy naturally as follows:

Entropy

$$H(X,Y) = E[-\log p(X,Y)]$$

and

$$H(X_1, X_2, \cdots, X_N) = E[-\log p(X_1, \cdots, X_N)]$$

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Conditional entropy

$$\begin{aligned} H(X,Y) &= E[-\log p(X,Y)] = E[-\log p(X) - \log p(Y|X)] \\ &= H(X) + \underbrace{E[-\log p(Y|X)]}_{H(Y|X)} \end{aligned}$$

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Interpretation

Total Info. of X and Y = Info. of X + Info. of Y knowing X

Expanding conditional entropy

$$H(Y|X) = E[-\log p(Y|X)]$$

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\begin{aligned}
H(Y|X) &= E[-\log p(Y|X)] \\
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The conditional entropy $H(Y|X)$ is essentially the average of $H(Y|x)$ over all possible value of x

Chain rule

Entropy

$$H(X_1, X_2, \cdots, X_N) = H(X_1) + H(X_2|X_1) + H(X_3|X_1, X_2) + \cdots + H(X_N|X_1, X_2, \cdots, X_{N-1}).$$

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Example

$$Pr(Rain, With\ umbrella) = 0.2 \qquad Pr(Rain, No\ umbrella) = 0.1$$
$$Pr(Sunny, With\ umbrella) = 0.2 \qquad Pr(Sunny, No\ umbrella) = 0.5$$

$$W \in \{Rain, Sunny\} \qquad U \in \{With\ umbrella, No\ umbrella\}$$

Entropies

$$H(W, U) = -0.2 \log 0.2 - 0.1 \log 0.1 - 0.2 \log 0.2 - 0.5 \log 0.5 = 1.76\ bits$$
$$H(W) = -0.3 \log 0.3 - 0.7 \log 0.7 = 0.88\ bits$$
$$H(U) = -0.4 \log 0.4 - 0.6 \log 0.6 = 0.97\ bits$$
$$H(W|U) = H(W, U) - H(U) = 0.79\ bits$$
$$H(U|W) = H(W, U) - H(W) = 0.88\ bits$$

Definition

It is often useful to gauge the difference between two distributions. KL-divergence is also known to be relative entropy. It is a way to measure the difference between two distributions. For two distributions of X, $p(x)$ and $p(y)$,

$$KL(p(x)\|q(x)) \triangleq \sum_{x\in\mathcal{X}} p(x)\log_2 \frac{p(x)}{q(x)}.$$

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- N.B. $KL(p(x)\|q(x)) \neq KL(q(x)\|p(x))$ in general

KL-divergence is non-negative

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KL(p(x)\|q(x)) &= \sum_{x\in\mathcal{X}} p(x)\log_2\frac{p(x)}{q(x)} \\
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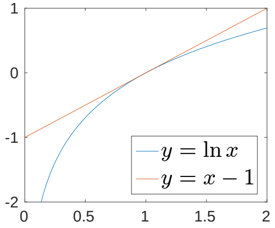

Fact

For any real x, $\ln(x) \le x - 1$. Moreover, the equality only holds when $x = 1$

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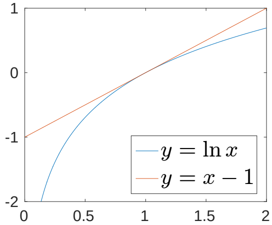

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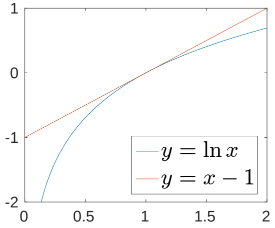

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We can define KL-divergence for continuous variables in a similar manner

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$$0 \leq KL(f \| \phi) = \int_x f(\mathbf{x}) \log \frac{f(\mathbf{x})}{\phi(\mathbf{x})} dx$$

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$$= \int_x f(\mathbf{x}) \log \phi(\mathbf{x}) dx$$

Application: Cross-entropy and cross-entropy error

In machine learning, it is often needed to assess the quality of a trained system. Consider the example of classifying an the political affliation of an individual

```
computed       | targets              | correct?
-----------------------------------------------
0.3  0.3  0.4  | 0  0  1 (democrat)   | yes
0.3  0.4  0.3  | 0  1  0 (republican) | yes
0.1  0.2  0.7  | 1  0  0 (other)      | no
```

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In a first glance, both examples appear to work equally well (or bad). Both have one classification error. However, a closer look will suggest the prediction of LHS is worse than RHS (why?)

(https://jamesmccaffrey.wordpress.com/2013/11/05/why-you-should-use-cross-entropy-error-instead-of-classification-error-or-mean-squared-error-for-neural-network-classifier-training/)

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----------------------------------------------
0.1  0.2  0.7   | 0  0  1 (democrat)    | yes
0.1  0.7  0.2   | 0  1  0 (republican)  | yes
0.3  0.4  0.3   | 1  0  0 (other)       | no
```

In a first glance, both examples appear to work equally well (or bad). Both have one classification error. However, a closer look will suggest the prediction of LHS is worse than RHS (why?) For a better assessment, we can treat both the computed result and the target result as distribution and compare them with KL-divergence. Namely

$$KL(p_{target}\|p_{computed}) = \sum_{group} p_{target}(group) \log \frac{p_{target}(group)}{p_{computed}(group)}$$

$$= -H(p_{target}) \underbrace{- \sum_{group} p_{target}(group) \log p_{computed}(group)}_{cross\ \ entropy}$$

(https://jamesmccaffrey.wordpress.com/2013/11/05/why-you-should-use-cross-entropy-error-instead-of-classification-error-or-mean-squared-error-for-neural-network-classifier-training/)

Application: Cross-entropy and cross-entropy error

$$\begin{aligned} Cross\ entropy(p\|q) &\triangleq \sum_x p(x)\log\frac{1}{q(x)} = E_p[-\log q(X)] \\ &= H(p) + KL(p\|q) \end{aligned}$$

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- To compute KL-divergence, one needs to find $H(p_{target})$, which is independent of the machine learning system and thus does not reflect the performance of the system
- Thus in practice, cross-entropy is commonly used instead of KL-divergence to measure the performance of a machine learning system

Application: Text processing

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- In text processing, it is common that one may need to measure the similiarity between two documents D_1 and D_2.
- How to represent documents? One may use the "bag of words". That is, to convert document into a vector of numbers. Each number is the count of a corresponding word
- One can then compares two documents using cross entropy

$$Cross\ entropy(p_1\|p_2) = \sum_w p_1(w) \log \frac{1}{p_2(w)},$$

where p_1 and p_2 are the word distributions of documents D_1 and D_2, respectively

Application: Text processing

It may be also interesting of comparing word distribution of a document to the word distribution across all documents That is, let q be the word distribution across all documents,

$$
\begin{aligned}
Cross\ entropy(p_1\|q) &= \sum_w p_1(w)\log\frac{1}{q(w)} \\
&= \sum_w \underbrace{\frac{\#\ w \text{ in } D_1}{\text{total } \#\text{ words in } D_1}\log\frac{\text{total } \#\text{ docs}}{\#\text{ doc with } w}}_{TF\text{-}IDF(w)},
\end{aligned}
$$

where $TF\text{-}IDF(w)$, short for term frequency-inverse document frequency, can reflect how important of the word w to the target document and can be used in search engine

Definition

As $H(X)$ is equivalent to the information revealed by X and $H(X|Y)$ the remaining information of X knowing Y, we expect that $H(X) - H(X|Y)$ is the information of X shared by $Y \Rightarrow$ "mutual information"

$$I(X;Y) \triangleq H(X) - H(X|Y)$$

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$$I(X;Y) \triangleq H(X) - H(X|Y)$$

Similarly, we can define the "conditional mutual information" shared between X and Y given Z as

$$I(X;Y|Z) \triangleq H(X|Z) - H(X|Y,Z)$$

Property of mutual information

$I(X;Y) = I(Y;X) \geq 0$

The definition is symmetric and non-negative as desired.

$$I(X;Y) = H(X) - H(X|Y) = E[-\log p(X)] - E[-\log p(X|Y)]$$

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&= \sum_{x,y} p(x,y)\log \frac{p(x,y)}{p(x)p(y)} = KL(p(x,y)\|p(x)p(y)) \geq 0
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&= \sum_{z} p(z) \sum_{x,y} p(x,y|z)\log \frac{p(x,y|z)}{p(x|z)p(y|z)} \\
&= \sum_{z} p(z) KL(p(x,y|z)\|p(x|z)p(y|z)) \geq 0
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Independence and mutual information

$I(X;Y)=0 \Leftrightarrow X\perp Y$

$$I(X;Y)=KL(p(x,y)\|p(x)p(y))=0$$

implies $p(x,y)=p(x)p(y)$. Therefore $X\perp Y$

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$I(X;Y|Z)=0 \Leftrightarrow X \perp Y|Z$

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implies $p(x,y|z)=p(x|z)p(y|z)$ for all z s.t. $p(z)>0$. Therefore $X\perp Y|Z$

Remark

This is just as what we expect. If there is no share information between X and Y, they should be indepedent!

Chain rule for mutual information

$$I(X_1, X_2, \cdots, X_N | Y)$$

Chain rule for mutual information

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\begin{aligned}
& I(X_1, X_2, \cdots, X_N | Y) \\
= & H(X_1, X_2, \cdots, X_N) - H(X_1, X_2, \cdots, X_N | Y)
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Mutual information for continuous variables

For continuous X, Y, Z, we can define $I(X;Y) = h(X) - h(X|Y)$ and $I(X;Y|Z) = h(X) - h(X|Y,Z)$
Then, the followings still hold true

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- $I(X;Y) = 0 \Leftrightarrow X \perp Y$
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- $I(X_1, X_2, \cdots, X_N|Y) = \sum_{i=1}^N I(X_i; Y|X^{i-1})$

Conditioning reduces entropy

Given more information, the residual information (uncertainty) should decrease.

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$$H(X) \geq H(X|Y) \qquad H(X|Y) \geq H(X|Y,Z)$$

This is obvious from our previous discussion since
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Of course, we also have

$$h(X) \geq h(X|Y) \qquad h(X|Y) \geq h(X|Y,Z)$$

since $h(X) - h(X|Y) = I(X;Y) \geq 0$ and
$h(X|Y) - h(X|Y) = I(X;Z|Y) \geq 0$

Data processing inequality

If random variables X, Y, Z satisfy $X \leftrightarrow Y \leftrightarrow Z$, then

$$I(X;Y) \geq I(X;Z).$$

Proof

$$I(X;Y) = I(X;Y,Z) - I(X;Z|Y)$$

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Proof

$$\begin{aligned} I(X;Y) &= I(X;Y,Z) - I(X;Z|Y) \\ &= I(X;Y,Z) \qquad\qquad (\text{since } X \leftrightarrow Y \leftrightarrow Z) \\ &= I(X;Z) + I(X;Y|Z) \\ &\geq I(X;Z) \end{aligned}$$

Application: perfect secrecy

Example (A simple cryptography example)

- Say you have a very personal letter that you don't want to let anyone else except some special someone to read

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Remark

Shannon's result: to ensure perfect secrecy, we can show that $H(M) \leq H(K)$

Application: perfect secrecy

Recall that M, C, K be plaintext message, ciphertext, and key, respectively

Assumption

We will assume here that we have a **non-probabilistic** *encryption scheme. In other words, each plaintext message maps to a unique ciphertext given a fixed key. So there is no ambiguity during decoding. Therefore,* $H(M|C,K) = 0$

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Remark (Independence)

For perfect secrecy, one should not be able to deduce anything regarding the message from the ciphertext. Therefore, C and M should be independent.

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$I(C; M) = 0 \Rightarrow H(M) = H(M|C) + I(C; M) = H(M|C)$

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Lemma (Entropy bound)

For any **non-probabilistic** *encryption scheme,* $H(M|C) \leq H(K|C)$

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We have perfect secrecy if $H(M) \leq H(K)$

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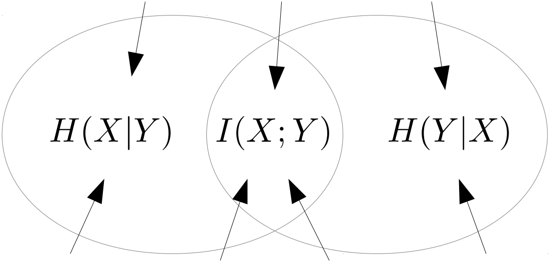
Proof.

Combine Corollary (Entropy bound) and Remark (Independence) □

Summary

$$H(X,Y) = H(X|Y) + I(X;Y) + H(Y|X)$$



$$H(X) = H(X|Y) + I(X;Y) \qquad I(X;Y) + H(Y|X) = H(Y)$$