Information Theory and Probabilistic Programming

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An optimization example

- Simple economy: m prosumers, n different goods[1]
- Each individual: production $\mathbf{p}_i \in \mathbb{R}_n$, consumption $\mathbf{c}_i \in \mathbb{R}_n$
- Expense of producing "$\mathbf{p}$" for agent $i = e_i(\mathbf{p})$
- Utility (happiness) of consuming "$\mathbf{c}$" units for agent $i = u_i(\mathbf{c})$
- Maximize happiness

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_{i=1}^{m} (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \qquad s.t. \qquad \sum_{i=1}^{m} \mathbf{c}_i = \sum_{i=1}^{m} \mathbf{p}_i$$

[1] Example borrowed from the first lecture of Prof Gordon's CMU CS 10-725

Walrasian equilibrium

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_{i=1}^{m} (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \qquad s.t. \qquad \sum_{i=1}^{m} \mathbf{c}_i = \sum_{i=1}^{m} \mathbf{p}_i$$

- Idea: introduce price λ_j to each good j. Let the market decide
  - Price $\lambda_j \uparrow$: consumption of good $j \downarrow$, production of good $j \uparrow$
  - Price $\lambda_j \downarrow$: consumption of good $j \uparrow$, production of good $j \downarrow$
  - Can adjust price until consumption = production for each good

Algorithm: tâtonnement

Assume that the appropriate prices are found, we can ignore the equality constraint, then the problem becomes

$$\max_{\mathbf{p}_i,\mathbf{c}_i} \sum_{i=1}^{m} (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i)) \quad \Rightarrow \quad \sum_{i=1}^{m} \max_{\mathbf{p}_i,\mathbf{c}_i} (u_i(\mathbf{c}_i) - e_i(\mathbf{p}_i))$$

So we can simply optimize production and consumption of each individual independently

Algorithm 1 tâtonnement

1: **procedure** FindBestPrices
2: $\lambda \leftarrow [0, 0, \cdots, 0]$
3: **for** $k = 1, 2, \cdots$ **do**
4: Each individual solves for its c_i and p_i for the given λ
5: $\lambda \leftarrow \lambda + \delta_k \sum_i (c_i - p_i)$

Lagrange multiplier

Problem

$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$g(\mathbf{x}) = 0$$

Consider $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$ and let $\tilde{f}(\mathbf{x}) = \min_\lambda L(\mathbf{x}, \lambda)$.

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$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) \text{ if } g(\mathbf{x}) = 0 \\ -\infty \text{ otherwise} \end{cases}$$

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Therefore, the problem is identical to $\max_{\mathbf{x}} \tilde{f}(\mathbf{x})$ or

$$\max_{\mathbf{x}} \min_{\lambda} (f(\mathbf{x}) - \lambda g(\mathbf{x})),$$

where λ is known to be the Lagrange multiplier.

Lagrange multiplier (con't)

Assume the optimum is a saddle point,

$$\max_{\mathbf{x}} \min_{\lambda} (f(\mathbf{x}) - \lambda g(\mathbf{x})) = \min_{\lambda} \max_{\mathbf{x}} (f(\mathbf{x}) - \lambda g(\mathbf{x})),$$

the R.H.S. implies

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$$

Inequality constraint

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$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$g(\mathbf{x}) \leq 0$$

Consider $\tilde{f}(\mathbf{x}) = \min_{\lambda \geq 0}(f(\mathbf{x}) - \lambda g(\mathbf{x}))$,

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Therefore, we can rewrite the problem as

$$\max_{\mathbf{x}} \min_{\lambda \geq 0}(f(\mathbf{x}) - \lambda g(\mathbf{x}))$$

Inequality constraint (con't)

Assume

$$\max_{\mathbf{x}} \min_{\lambda \geq 0} (f(\mathbf{x}) - \lambda g(\mathbf{x})) = \min_{\lambda \geq 0} \max_{\mathbf{x}} (f(\mathbf{x}) - \lambda g(\mathbf{x}))$$

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Moreover, at the optimum point $(\mathbf{x}^*, \lambda^*)$, we should have the so-called "complementary slackness" condition

$$\lambda^* g(\mathbf{x}^*) = 0$$

since

$$\max_{\substack{\mathbf{x} \\ g(\mathbf{x}) \leq 0}} f(\mathbf{x}) \equiv \max_{\mathbf{x}} \min_{\lambda \geq 0} (f(\mathbf{x}) - \lambda g(\mathbf{x}))$$

Karush-Kuhn-Tucker conditions

Problem

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$g(\mathbf{x}) \le 0, \quad h(\mathbf{x}) = 0$$

Conditions

$$\begin{aligned}
\nabla f(\mathbf{x}^*) - \mu^* \nabla g(\mathbf{x}^*) - \lambda^* \nabla h(\mathbf{x}^*) &= 0 \\
g(\mathbf{x}^*) &\le 0 \\
h(\mathbf{x}^*) &= 0 \\
\mu^* &\ge 0 \\
\mu^* g(\mathbf{x}^*) &= 0
\end{aligned}$$

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- However, we want to make sure that we can losslessly decode the message also!

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  - We say $c(\mathbf{x})$ is uniquely decodable if all input sequences map to different outputs

Prefix-free code

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- Instead, for a mapping $a \mapsto 1, b \mapsto 01, c \mapsto 001, d \mapsto 0001$, I will argue that we can always decode a symbol "once it is available"
  - Note that the catch is that there is no codeword being the "prefix" of another codeword
  - We call such code a prefix-free code or an instantaneous code

Kraft's Inequality

- How do we know if a length profile for a code is possible?
- Kraft's inequality: Consider a length profile $l_1, l_2, \cdots, l_K$, there exists a uniquely decodable code for symbols $x_1, x_2, \cdots, x_K$ such that $l(x_1) = l_1$, $l(x_2) = l_2, \cdots, l(x_K) = l_K$ if and only if $\sum_{k=1}^{K} 2^{-l_k} \leq 1$

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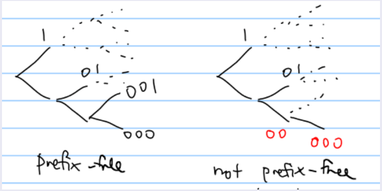
Intuition

Consider # "descendants" of each codeword at the "l_{max}"-level, then for prefix-free code, we have

$$\sum_{k=1}^{K} 2^{l_{max}-l} \leq 2^{l_{max}}$$



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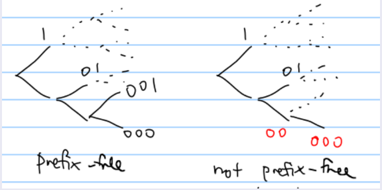

Forward Proof

Given $l_1, l_2, \cdots, l_K$ satisfy $\sum_{k=1}^{K} 2^{-l_k} \leq 1$, we can assign nodes on a tree as previous slides. More precisely,

- Assign i-th node as a node at level l_i, then cross out all its descendants
- Repeat the procedure for i from 1 to K
- We know that there are sufficient tree nodes to be assigned since the Kraft's inequaltiy is satisfied

The corresponding code is apparently prefix-free and thus is uniquely decodable

Converse Proof

Consider message from coding k symbols $\mathbf{x} = x_1, x_2, \cdots, x_k$

$$
\begin{aligned}
\left(\sum_{x\in\mathcal{X}} 2^{-l(x)}\right)^k &= \left(\sum_{x_1\in\mathcal{X}} 2^{-l(x_1)}\right)\left(\sum_{x_2\in\mathcal{X}} 2^{-l(x_2)}\right)\cdots\left(\sum_{x_k\in\mathcal{X}} 2^{-l(x_k)}\right) \\
&= \sum_{x_1,x_2,\cdots,x_k\in\mathcal{X}^k} 2^{-(l(x_1)+l(x_2)+\cdots+l(x_k))} \\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} 2^{-l(\mathbf{x})}
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$$= \sum_{x_1,x_2,\cdots,x_k\in\mathcal{X}^k} 2^{-(l(x_1)+l(x_2)+\cdots+l(x_k))}$$

$$= \sum_{\mathbf{x}\in\mathcal{X}^k} 2^{-l(\mathbf{x})} = \sum_{m=1}^{kl_{max}} a(m)2^{-m},$$

where $a(m)$ is the number of codeword with length m. However, for the code to be uniquely decodable, $a(m) \leq 2^m$, where 2^m is the number of available codewords with length m.

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$$\sum_{x\in\mathcal{X}} 2^{-l(x)} \leq (kl_{max})^{1/k} \approx 1 \text{ as } k \to \infty$$

Minimum rate required to compress a source

$$min_{l_1,l_2,\cdots,l_K} \sum_{k=1}^{K} p_k l_k \text{ subject to } \sum_{k=1}^{K} 2^{-l_k} \le 1 \text{ and } l_1,\cdots,l_K \ge 0$$

$$\equiv max_{l_1,l_2,\cdots,l_K} - \sum_{k=1}^{K} p_k l_k \text{ subject to } \sum_{k=1}^{K} 2^{-l_k} - 1 \le 0 \text{ and } -l_1,\cdots,-l_K \le 0$$

KKT conditions

$$-\nabla\left(\sum_{k=1}^{K} p_k l_k\right) - \mu_0 \nabla\left(\sum_{k=1}^{K} 2^{-l_k} - 1\right) + \sum_{k=1}^{K} \mu_k \nabla l_k = 0$$

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$$\mu_0\left(\sum_{k=1}^{K} 2^{-l_k} - 1\right) = 0, \quad \mu_k l_k = 0$$

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And by $\sum_{k=1}^{K} 2^{-l_k} \leq 1$, we have

$$\sum_{k=1}^{K} \frac{p_j}{\mu_0 \log 2} = \frac{1}{\mu_0 \log 2} \leq 1 \Rightarrow \mu_0 \geq \frac{1}{\log 2}$$

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Note that as $\mu_0 \downarrow$, $\frac{p_j}{\mu_0 \log 2} \uparrow$ and $l_j \downarrow$.

Minimum rate required to compress a source

Since we expect $l_k > 0$, $\mu_k = 0$. Expand the first equation, we get

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Note that as $\mu_0 \downarrow$, $\frac{p_j}{\mu_0 \log 2} \uparrow$ and $l_j \downarrow$. Therefore, if we want to decrease code rate, we should reduce μ_0 as much as possible. Thus, take $\mu_0 = \frac{1}{\log 2}$. Then $2^{-l_j} = p_j \Rightarrow l_j = -\log_2 p_j$. Thus, the minimum rate becomes

$$\sum_{k=1}^{K} p_k l_k = -\sum_{k=1}^{K} p_k \log_2 p_k \triangleq H(p_1, \cdots, p_K)$$

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  - We cannot compress a source losslessly below its entropy
  - On the other hand, since Kraft's inequality guarantee existence of code, we should be able to find code to achieve the theoretical limit
- However, the proof we discussed was not constructive. How can we actually design a code to compress arbitrarily close to the theoretical limit?

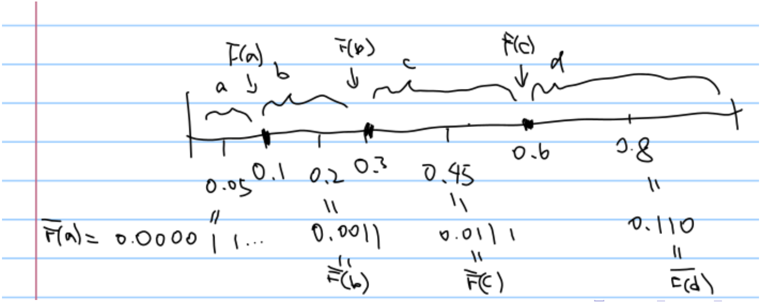
Shannon-Fano-Elias code

Key idea

Each codeword corresponds to an intervel of $[0,1]$

Example

110 corresponds to $[0.110, 0.110\dot{1}] = [0.11, 0.111) = [0.75, 0.875)$



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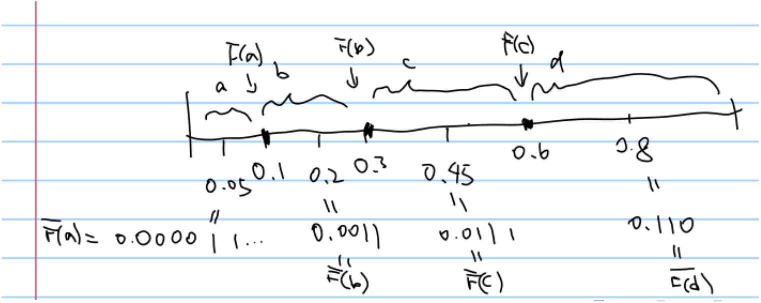

Observations

Remark (Observation 1)

Let $l(x) = |c(x)|$ *be the length of the SFE codeword, and let* $u(x)$ *be the corresponding interval. Then, the length of the interval* $|u(x)| = 2^{-l(x)}$

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Proof of Observation 2.

$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$. We will show instead if $c(x_1)$ and $c(x_2)$ are prefix of one another, then $u(x_1)$ and $u(x_2)$ overlap. WLOG, assume $c(x_1)$ is a prefix of $c(x_2)$, the lower boundary of $u(x_1)$ is below the lower boundary of $u(x_2)$ and yet the upper boundary of $u(x_1)$ is above the upper boundary of $u(x_2)$. Thus, $u(x_2) \subseteq u(x_1)$ and hence $u(x_1)$ and $u(x_2)$ overlap each other $\square$

Example

Consider a source that
$p(x_1) = 0.25, p(x_2) = 0.25, p(x_3) = 0.2, p(x_4) = 0.15, p(x_5) = 0.15$

x	$p(x)$	$F(x)$	$\overline{F}(x)$	$\overline{F}(x)$ in Binary	$l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil + 1$	Codeword
1	0.25	0.25	0.125	0.001	3	001
2	0.25	0.5	0.375	0.011	3	011
3	0.2	0.7	0.6	$0.1\overline{0011}$	4	1001
4	0.15	0.85	0.775	$0.110\overline{0011}$	4	1100
5	0.15	1.0	0.925	$0.111\overline{0110}$	4	1110

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  - Since no codeword can overlap in SFE, no code word can be prefix of another
- Average code rate is upper bounded by $H(X) + 2$

$$\sum_{x\in\mathcal{X}} p(x) l(x) = \sum_{x\in\mathcal{X}} p(x) \left(\left\lceil \log_2 \frac{1}{p(x)} \right\rceil + 1 \right)$$
$$\leq \sum_{x\in\mathcal{X}} p(x) \left(\log_2 \frac{1}{p(x)} + 2 \right) = H(X) + 2$$

"Symbol grouping" trick

- Let's consider two symbols as a super-symbol and compress the pair at each time with SFE code
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&= -\sum_{x_1\in\mathcal{X}} p(x_1)\log_2 p(x_1) - \sum_{x_2\in\mathcal{X}} p(x_2)\log_2 p(x_2)\\
&= 2H(X)
\end{aligned}$$

Therefore, the code rate per original symbol is upper bounded by

$$\frac{1}{2}(H(X_S)+2) = H(X)+1$$

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Therefore as long as a given rate $R>H(X)$, we can always find a large enough N such that the code rate using the "grouping trick" and SFE code is below R. This concludes the forward proof

Entropy for discrete random variable

Von Neumman to Shannon

"You should call it entropy for two reasons: first because that is what the formula is in statistical mechanics but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage!" -John von Neumman

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- A less likely event has "more" information and requires more bits to store. $H(X)$ is just the average number of bits required

Biased coin with $Pr(Head) = p$

$$H(X) = -Pr(Head)\log Pr(Head) - Pr(Tail)\log Pr(Tail)$$
$$= -p\log p - (1-p)\log(1-p)$$

- Entropy is largest (=1) when $p = 0.5$
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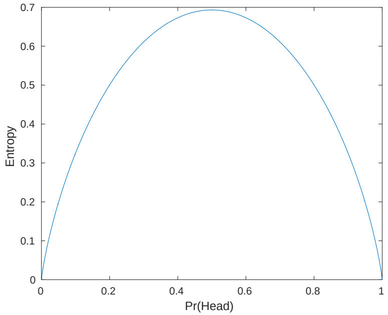

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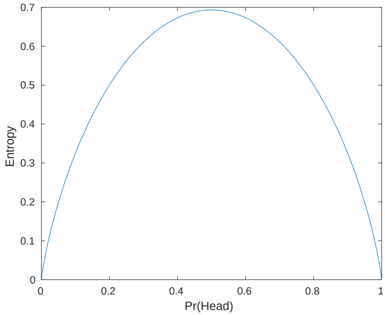
- Entropy can be interpreted as *the average uncertainty of the outcome* or *the amount of information "gained" after the outcome is revealed*



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The definition makes little sense for a continuous X. Since the probability of an outcome x is always 0, we may define instead the differential entropy for X as

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Differential entropy of common distributions

Uniform Distribution

If $p(X) = \begin{cases} 1/a & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$

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For exponentially distributed $T \sim Exp(\lambda)$,

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N.B. $h(X)$ only depends on σ^2 and is independent of μ as one would expect

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For N-dim multivariate normal distributed $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$,

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$$= -E\left[\log\left(\frac{1}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right)\right)\right]$$

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&= -E\left[\log\left(\frac{1}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right)\right)\right] \\
&= \log\sqrt{\det(2\pi\Sigma)} + \frac{\log e}{2}E\left[\sum_{i,j}(X_i-\mu_i)\left[\Sigma^{-1}\right]_{i,j}(X_j-\mu_j)\right] \\
&= \log\sqrt{\det(2\pi\Sigma)} + \frac{\log e}{2}\sum_{i,j}\left[\Sigma^{-1}\right]_{i,j}E\left[(X_j-\mu_j)(X_i-\mu_i)\right] \\
&= \log\sqrt{\det(2\pi\Sigma)} + \frac{\log e}{2}\sum_{i,j}\left[\Sigma^{-1}\right]_{i,j}\Sigma_{j,i} \\
&= \log\sqrt{\det(2\pi\Sigma)} + \frac{N\log e}{2} = \log\sqrt{e^N\det(2\pi\Sigma)} = \log\sqrt{\det(2\pi e\Sigma)}
\end{aligned}$$

Differential entropy and entropy

How differential entropy is related to its discrete counterpart?

- Consider a continuous random variable X
- Let X^{Δ} is a "quantized" version of it with quantization stepsize of Δ

$$H(X^{\Delta}) = \sum -p_{X^{\Delta}}(x^{\Delta}) \log p_{X^{\Delta}}(x^{\Delta})$$

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&= h(X) - \log\Delta
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Consider the processing time of a packet follow an exponential distribution with an average of 1 ms. If we want to store the time with the precision of 0.01 ms, about how many bits are needed to store the result?

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- The corresponding differential entropy $h(T) = 1 - \log(\lambda) = 1$
- If we want to store with precision of 0.01 ms, we need $h(T) - \log 0.01 \approx 7.64bits$

Lower bound of entropy

$H(X) \geq 0$

Since $p(X) \leq 1$, $-\log p(X) \geq 0$, therefore

$$H(X) = E[-\log p(X)] \geq 0$$

After all, $H(X)$ represents the required bits to compress the source X

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Caveat

It does NOT need to be true for differential entropy. It is possible that

$$h(X) < 0$$

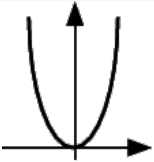
For example, for a uniformly distributed X from 0 to 0.5,

$$h(X) = \log 0.5 = -1$$

Jensen's Inequality

For a convex (bowl-shape) function f

$$E[f(X)] \geq f(E[X])$$



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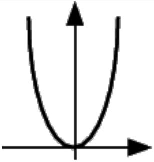

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Let us consider X with only two outcomes x_1 and x_2 with probabilities p and $1-p$. Easy to see that

$$E[f(X)] = pf(x_1) + (1-p)f(x_2) \geq f(px_1 + (1-p)x_2) = f(E[X])$$

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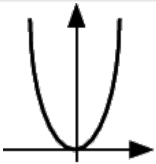

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Result can be extended to variables with more than two outcomes easily

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Examples

You should know this bound long alone. Think of the maximum number of bits needed:

- to store the outcome of flipping a coin: $\log 2 = 1$ bit
- to store the outcome of throwing a dice: $\log 6 \leq 3$ bits

Review

- Source coding theorem: For an independent and identically distributed (i.i.d.) discrete memoryless source (DMS) X, we can always compress it with no less than $H(X)$ bits per input symbol, where $H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x) = E[-\log p(X)]$

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- For a quantized version of continuous X, $H(X_\Delta) = h(X) - \log\Delta$
- For multivariate normal $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$,

$$h(\boldsymbol{X}) = \log\sqrt{\det(2\pi e\Sigma)}$$

Upper bound of differential entropy

$$h(X) \leq \log E\left[\frac{1}{p(X)}\right] = \log \int_{x \in \mathcal{X}} p(x) \frac{1}{p(x)} dx = \log |\mathcal{X}|$$

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- The expression still makes sense but it is not useful usually since the sampling space can be unbounded $|\mathcal{X}| = \infty$ (for example, normally distributed X)
- Thus it makes much more sense to consider upper bound of a differential entropy constrained on the variance of the variable (why not constrained on mean?)
- It turns out that for a fixed variance σ^2, the variable will have largest differential entropy if it is normally distributed (will show later). Thus

$$h(X) \leq \log\sqrt{2\pi e\sigma^2}$$

Joint entropy

For multivariate random variable, we can extend the definition of entropy naturally as follows:

Entropy

$$H(X, Y) = E[-\log p(X, Y)]$$

and

$$H(X_1, X_2, \cdots, X_N) = E[-\log p(X_1, \cdots, X_N)]$$

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Conditional entropy

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\begin{aligned}
H(X,Y) &= E[-\log p(X,Y)] = E[-\log p(X) - \log p(Y|X)] \\
&= H(X) + \underbrace{E[-\log p(Y|X)]}_{H(Y|X)}
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Interpretation

Total Info. of X and Y = Info. of X + Info. of Y knowing X

Expanding conditional entropy

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The conditional entropy $H(Y|X)$ is essentially the average of $H(Y|x)$ over all possible value of x

Chain rule

Entropy

$$H(X_1, X_2, \cdots, X_N) = H(X_1) + H(X_2|X_1) + H(X_3|X_1, X_2) + \cdots + H(X_N|X_1, X_2, \cdots, X_{N-1}).$$

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Example

$$Pr(Rain, With\ umbrella) = 0.2 \qquad Pr(Rain, No\ umbrella) = 0.1$$
$$Pr(Sunny, With\ umbrella) = 0.2 \qquad Pr(Sunny, No\ umbrella) = 0.5$$

$$W \in \{Rain, Sunny\} \qquad U \in \{With\ umbrella, No\ umbrella\}$$

Entropies

$$H(W,U) = -0.2\log 0.2 - 0.1\log 0.1 - 0.2\log 0.2 - 0.5\log 0.5 = 1.76\ bits$$
$$H(W) = -0.3\log 0.3 - 0.7\log 0.7 = 0.88\ bits$$
$$H(U) = -0.4\log 0.4 - 0.6\log 0.6 = 0.97\ bits$$
$$H(W|U) = H(W,U) - H(U) = 0.79\ bits$$
$$H(U|W) = H(W,U) - H(W) = 0.88\ bits$$

Definition

It is often useful to gauge the difference between two distributions. KL-divergence is also known to be relative entropy. It is a way to measure the difference between two distributions. For two distributions of X, $p(x)$ and $p(y)$,

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KL-divergence is non-negative

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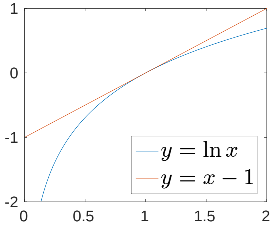

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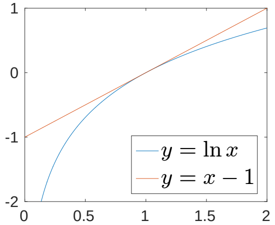

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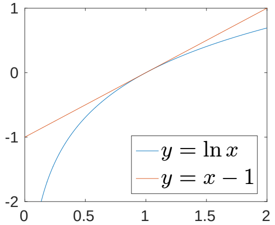

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We can define KL-divergence for continuous variables in a similar manner

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Let's consider the multivariate case with a fixed covariance matrix Σ, the univariate (scalar) case is a special case thus automatically taken care of. Without loss of generality, let's consider zero mean. Denote $\mathcal{N}(\mathbf{x}; \mathbf{0}, \Sigma) = \phi(\mathbf{x})$.

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Proof

Let's consider the multivariate case with a fixed covariance matrix Σ, the univariate (scalar) case is a special case thus automatically taken care of. Without loss of generality, let's consider zero mean. Denote $\mathcal{N}(\mathbf{x};\mathbf{0},\Sigma)=\phi(\mathbf{x})$. For any other distribution $f(\mathbf{x})$ with the same covariance matrix Σ, first note that $\int_{\mathbf{x}} f(\mathbf{x})\log\phi(\mathbf{x})d\mathbf{x}=\int_{\mathbf{x}}\phi(\mathbf{x})\log\phi(\mathbf{x})d\mathbf{x}$ (to be show in the next slide).

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$$0 \leq KL(f \| \phi) = \int_{\mathbf{x}} f(\mathbf{x}) \log \frac{f(\mathbf{x})}{\phi(\mathbf{x})} d\mathbf{x}$$

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$$

Application: Thiel index

- Measure economic inequality among different groups or for a group of individuals
- Let p_i be the economic wealth proportion of group i, and q_i be the population size proportion of group i
- Thiel index is simply $KL(p||q)$
- Let's apply to a group of N individuals.
    - If they all have the same wealth, both p and q are uniform ($p_i = q_i = 1/N$), thus Thiel index $= KL(p||q) = 0$
    - If one of them own everything, q is uniform but p is a δ-function. Thus Thiel index $= KL(p||q) = \sum_i p_i \log \frac{p_i}{q_i} = \log \frac{1}{1/N} = \log N$

Application: Cross-entropy and cross-entropy error

In machine learning, it is often needed to assess the quality of a trained system. Consider the example of classifying an the political affliation of an individual

```
computed       | targets              | correct?
-----------------------------------------------
0.3  0.3  0.4  | 0  0  1 (democrat)   | yes
0.3  0.4  0.3  | 0  1  0 (republican) | yes
0.1  0.2  0.7  | 1  0  0 (other)      | no
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In a first glance, both examples appear to work equally well (or bad). Both have one classification error. However, a closer look will suggest the prediction of LHS is worse than RHS (why?)

(https://jamesmccaffrey.wordpress.com/2013/11/05/why-you-should-use-cross-entropy-error-instead-of-classification-error-or-mean-squared-error-for-neural-network-classifier-training/)

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In a first glance, both examples appear to work equally well (or bad). Both have one classification error. However, a closer look will suggest the prediction of LHS is worse than RHS (why?) For a better assessment, we can treat both the computed result and the target result as distribution and compare them with KL-divergence. Namely

$$KL(p_{target}\|p_{computed}) = \sum_{group} p_{target}(group) \log \frac{p_{target}(group)}{p_{computed}(group)}$$

$$= -H(p_{target}) \underbrace{- \sum_{group} p_{target}(group) \log p_{computed}(group)}_{cross\ \ entropy}$$

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\begin{aligned}
Cross\ entropy(p\|q) &\triangleq \sum_x p(x)\log\frac{1}{q(x)} = E_p[-\log q(X)] \\
&= H(p) + KL(p\|q)
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- To compute KL-divergence, one needs to find $H(p_{target})$, which is independent of the machine learning system and thus does not reflect the performance of the system
- Thus in practice, cross-entropy is commonly used instead of KL-divergence to measure the performance of a machine learning system

Example: Text processing

- In text processing, it is common that one may need to measure the similiarity between two documents D_1 and D_2.

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Example: Text processing

- In text processing, it is common that one may need to measure the similiarity between two documents D_1 and D_2.
- How to represent documents? One may use the "bag of words". That is, to convert document into a vector of numbers. Each number is the count of a corresponding word
- One can then compares two documents using cross entropy

$$Cross\ entropy(p_1 \| p_2) = \sum_w p_1(w) \log \frac{1}{p_2(w)},$$

where p_1 and p_2 are the word distributions of documents D_1 and D_2, respectively

TF-IDF and cross entropy

It may be also interesting of comparing word distribution of a document to the word distribution across all documents That is, let q be the word distribution across all documents,

$$\begin{aligned} Cross\ entropy(p_1\|q) &= \sum_w p_1(w)\log\frac{1}{q(w)} \\ &= \sum_w \underbrace{\frac{\#\ w \text{ in } D_1}{\text{total } \#\text{ words in } D_1}\log\frac{\text{total } \#\text{ docs}}{\#\text{ doc with } w}}_{TF\text{-}IDF(w)}, \end{aligned}$$

where $TF\text{-}IDF(w)$, short for term frequency-inverse document frequency, can reflect how important of the word w to the target document and can be used in search engine

Definition

As $H(X)$ is equivalent to the information revealed by X and $H(X|Y)$ the remaining information of X knowing Y, we expect that $H(X) - H(X|Y)$ is the information of X shared by $Y \Rightarrow$ "mutual information"

$$I(X;Y) \triangleq H(X) - H(X|Y)$$

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$$I(X;Y) \triangleq H(X) - H(X|Y)$$

Similarly, we can define the "conditional mutual information" shared between X and Y given Z as

$$I(X;Y|Z) \triangleq H(X|Z) - H(X|Y,Z)$$

Property of mutual information

$I(X;Y) = I(Y;X) \geq 0$

The definition is symmetric and non-negative as desired.

$$I(X;Y) = H(X) - H(X|Y) = E[-\log p(X)] - E[-\log p(X|Y)]$$

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&= \sum_{z} p(z) \sum_{x,y} p(x,y|z)\log \frac{p(x,y|z)}{p(x|z)p(y|z)} \\
&= \sum_{z} p(z) KL(p(x,y|z) \| p(x|z)p(y|z)) \geq 0
\end{aligned}
$$

Independence and mutual information

$I(X;Y)=0 \Leftrightarrow X \perp Y$

$$I(X;Y)=KL(p(x,y)\|p(x)p(y))=0$$

implies $p(x,y)=p(x)p(y)$. Therefore $X \perp Y$

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$I(X;Y|Z)=0 \Leftrightarrow X \perp Y|Z$

$$I(X;Y|Z)=\sum_z p(z)KL(p(x,y|z)\|p(x|z)p(y|z))=0$$

implies $p(x,y|z)=p(x|z)p(y|z)$ for all z s.t. $p(z)>0$. Therefore $X \perp Y|Z$

Remark

This is just as what we expect. If there is no share information between X and Y, they should be indepedent!

Chain rule for mutual information

$$I(X_1, X_2, \cdots, X_N | Y)$$

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\begin{aligned}
&I(X_1, X_2, \cdots, X_N|Y) \\
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N.B. $X^N = X_1, X_2, \cdots, X_N$

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Mutual information for continuous variables

For continuous X, Y, Z, we can define $I(X;Y) = h(X) - h(X|Y)$ and
$I(X;Y|Z) = h(X|Z) - h(X|Y,Z)$
Then, the followings still hold true

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- $I(X;Y|Z) = 0 \Leftrightarrow X \perp Y|Z$
- $I(X_1, X_2, \cdots, X_N|Y) = \sum_{i=1}^{N} I(X_i; Y|X^{i-1})$

Conditioning reduces entropy

Given more information, the residual information (uncertainty) should decrease.

Conditioning reduces entropy

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$$H(X) \geq H(X|Y) \qquad H(X|Y) \geq H(X|Y,Z)$$

This is obvious from our previous discussion since
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Of course, we also have

$$h(X) \geq h(X|Y) \qquad h(X|Y) \geq h(X|Y,Z)$$

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Data processing inequality

If random variables X, Y, Z satisfy $X \leftrightarrow Y \leftrightarrow Z$, then

$$I(X;Y) \geq I(X;Z).$$

Proof

$$I(X;Y) = I(X;Y,Z) - I(X;Z|Y)$$

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Proof

$$\begin{aligned} I(X;Y) &= I(X;Y,Z) - I(X;Z|Y) \\ &= I(X;Y,Z) \qquad \text{(since } X \leftrightarrow Y \leftrightarrow Z\text{)} \\ &= I(X;Z) + I(X;Y|Z) \\ &\geq I(X;Z) \end{aligned}$$

Application: perfect secrecy

Example (A simple cryptography example)

- Say you have a very personal letter that you don't want to let anyone else except some special someone to read

Application: perfect secrecy

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  - Code: ciphertext C
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Remark

Shannon's result: to ensure perfect secrecy, we can show that $H(M) \leq H(K)$

Application: perfect secrecy

Recall that M, C, K be plaintext message, ciphertext, and key, respectively

Assumption

We will assume here that we have a ***non-probabilistic*** *encryption scheme. In other words, each plaintext message maps to a unique ciphertext given a fixed key. So there is no ambiguity during decoding. Therefore,* $H(M|C, K) = 0$

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Remark (Independence)

For perfect secrecy, one should not be able to deduce anything regarding the message from the ciphertext. Therefore, C and M should be independent.

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Remark (Independence)

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$I(C;M) = 0 \Rightarrow H(M) = H(M|C) + I(C;M) = H(M|C)$

Application: perfect secrecy

Lemma (Entropy bound)

For any ***non-probabilistic*** *encryption scheme,* $H(M|C) \leq H(K|C)$

Application: perfect secrecy

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Proof.

Recall that for non-probabilistic encryption scheme, $H(M|K, C) = 0 \Rightarrow$ $H(M|C) \leq H(M, K|C)$

Application: perfect secrecy

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Application: perfect secrecy

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For any non-probabilistic encryption scheme, $H(M|C) \le H(K)$

Theorem (Perfect secrecy)

We have perfect secrecy if $H(M) \le H(K)$

Application: perfect secrecy

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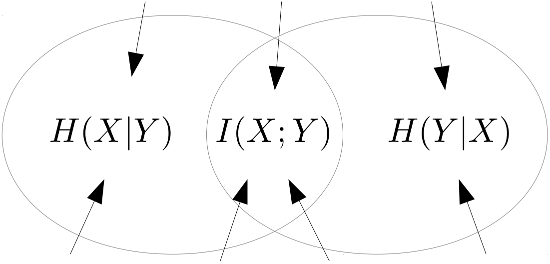
Proof.

Combine Corollary (Entropy bound) and Remark (Independence) □

Summary

$$H(X,Y) = H(X|Y) + I(X;Y) + H(Y|X)$$



$$H(X) = H(X|Y) + I(X;Y) \qquad I(X;Y) + H(Y|X) = H(Y)$$

Review

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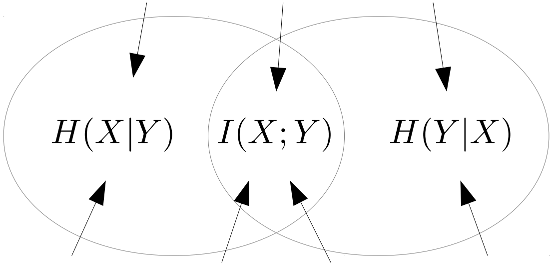

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Review

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- Independence and mutual information:
  - $X \perp Y \Leftrightarrow$

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- Independence and mutual information:
  - $X \perp Y \Leftrightarrow I(X;Y) = 0$
  - $X \perp Y|Z \Leftrightarrow$

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- KL-divergence: $KL(p||q) \triangleq \sum_x p(x) \log \frac{p(x)}{q(x)} \geq 0$

This time

- Identification/Decision trees
- Random forests
- Law of Large Number
- Asymptotic equipartition (AEP) and typical sequences

Vampire database

Romanian Data Base

Vampire?	Shadow?	Garlic?	Complexion?	Accent?
No	?	Yes	Pale	None
No	Yes	Yes	Ruddy	None
Yes	?	No	Ruddy	None
Yes	No	No	Average	Heavy
Yes	?	No	Average	Odd
No	Yes	No	Pale	Heavy
No	Yes	No	Average	Heavy
No	?	Yes	Ruddy	Odd

(https://www.youtube.com/watch?v=SXBG3RGr_Rc)

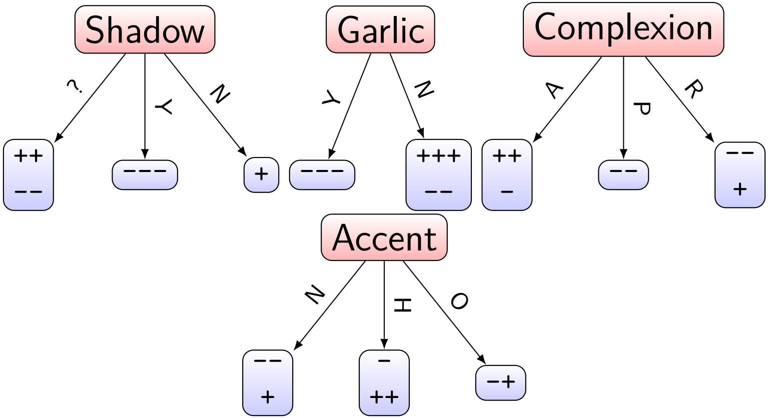
Identifying vampire

Goal: Design a set of tests to identify vampires

Potential difficulties

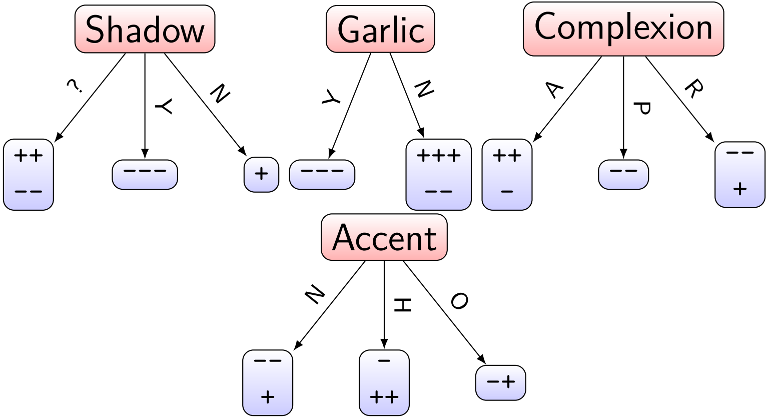
- Non-numerical data
- Some information may not matter
- Some may matter only sometimes
- Tests may be costly $\Rightarrow$ conduct as few as possible

Test trees



$+$: Vampire $\quad -$: Not vampire

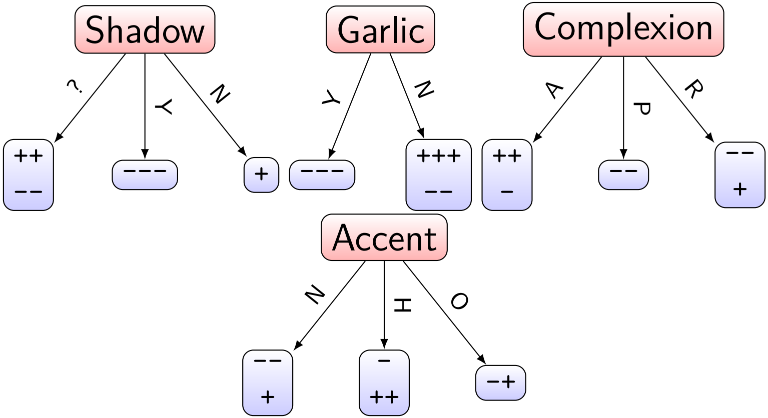
Test trees



+ : Vampire − : Not vampire

How to pick a good test?

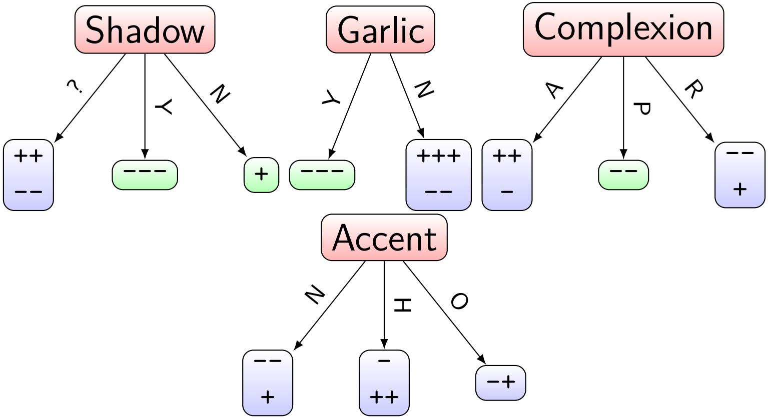
Test trees



+ : Vampire − : Not vampire

How to pick a good test? Pick test that identifies most vampires (and non-vampires)!

Sizes of homogeneous sets



$+$: Vampire $\quad -$: Not vampire

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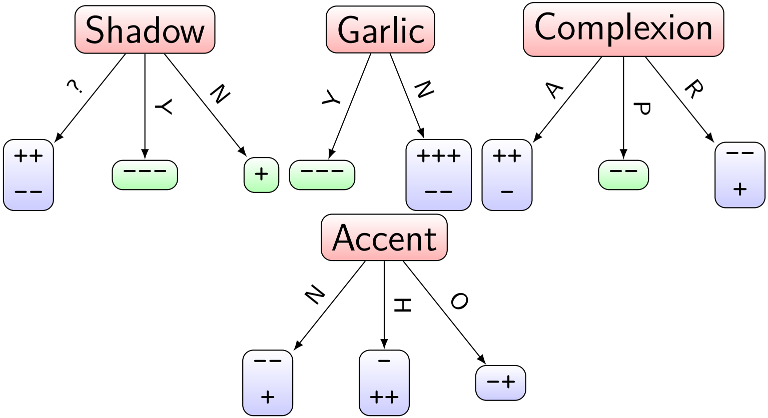
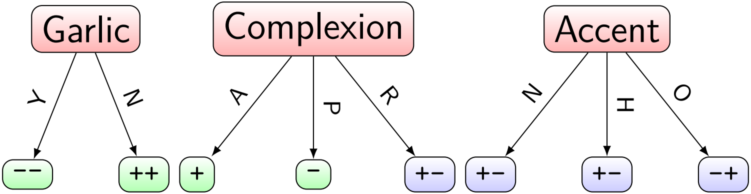


$+$: Vampire $-$: Not vampire

Shadow: 4 Garlic: 3 Complexion: 2 Accent: 0

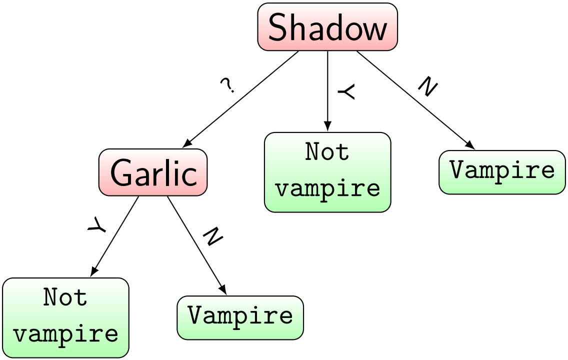
Picking second test

Let say we pick "shadow" as the first test after all. Then, for the remaining unclassified individuals,



Garlic: 4 Complexion: 2 Accent: 0

Combined tests



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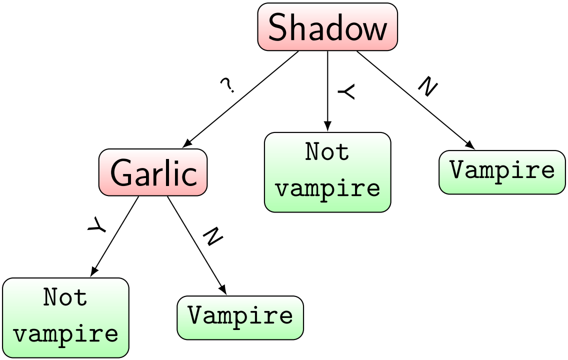
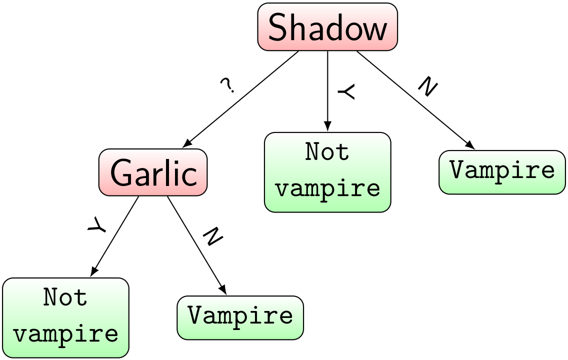


Problem

When our database size increases, none of the test likely to completely separate vampire from non-vampire. All tests will score 0 then.

Combined tests



Problem

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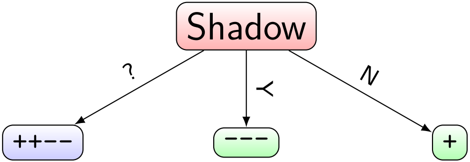
Entropy comes to the rescue!

Conditional entropy as a measure of test efficiency

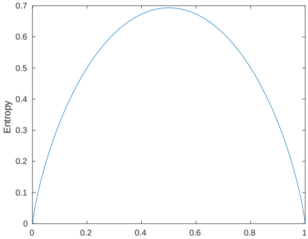
Consider the database is randomly sampled from a distribution. A set is

- Very homogeneous $\approx$ high certainty
- Not so homogenous $\approx$ high randomness

These can be measured with its entropy



$H(V|S=?)=1$ $H(V|S=Y)=0$ $H(V|S=N)=0$



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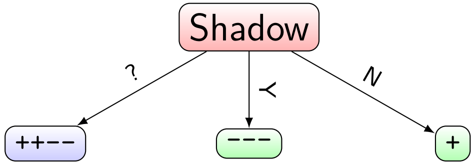

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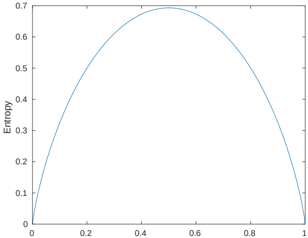

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)$$

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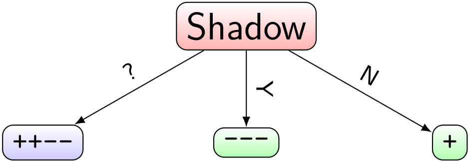

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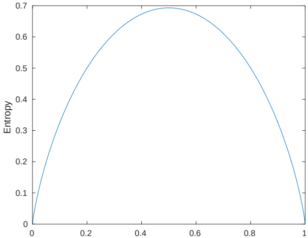

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)$$

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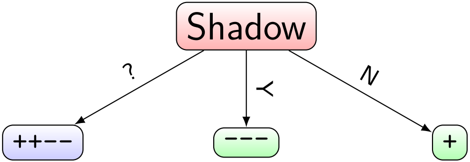
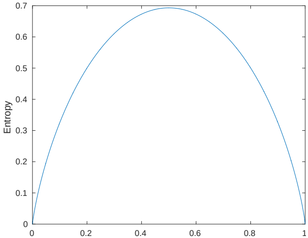


$H(V|S=?)=1$ $H(V|S=Y)=0$ $H(V|S=N)=0$



Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)+\frac{1}{8}H(V|S=N)=0.5$$

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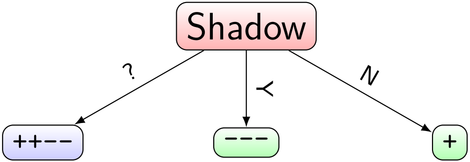

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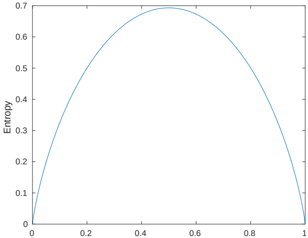

Remaining uncertainty given the test:

$$\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)+\frac{1}{8}H(V|S=N)=0.5$$
$$=Pr(S=?)H(V|S=?)+Pr(S=Y)H(V|S=Y)+Pr(S=N)H(V|S=N)$$

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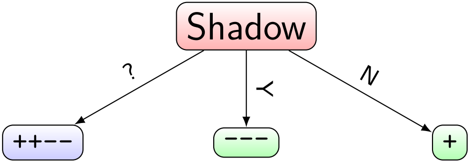

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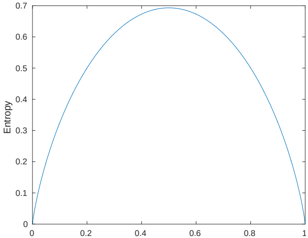
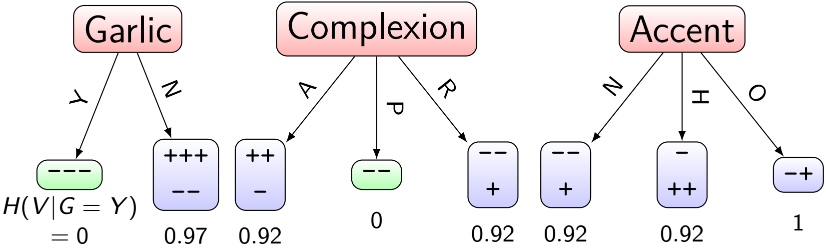


Remaining uncertainty given the test:

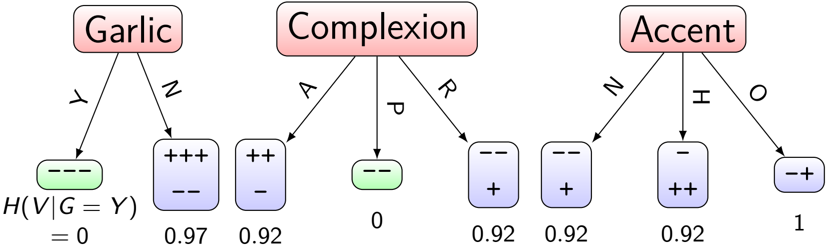
$$
\begin{aligned}
&\frac{4}{8}H(V|S=?)+\frac{3}{8}H(V|S=Y)+\frac{1}{8}H(V|S=N)=0.5\\
=&Pr(S=?)H(V|S=?)+Pr(S=Y)H(V|S=Y)+Pr(S=N)H(V|S=N)\\
=&H(V|S)
\end{aligned}
$$

Remaining uncertainty



$H(V|S) = 0.5$

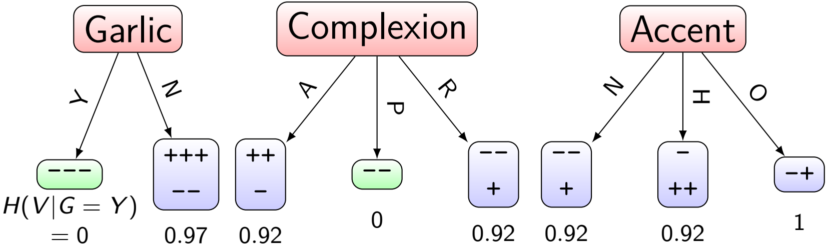
Remaining uncertainty



$$H(V|S) = 0.5$$

$$H(V|G) = \frac{3}{8}\cdot 0 + \frac{5}{8}\cdot 0.97 = 0.61$$

Remaining uncertainty



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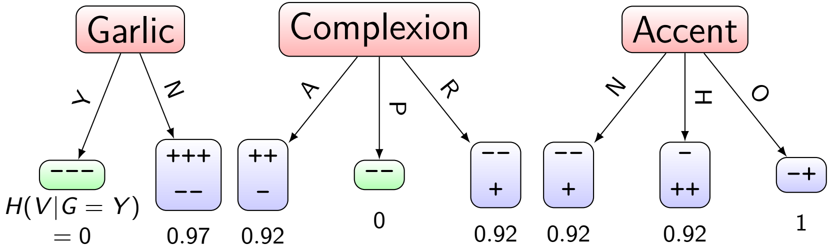

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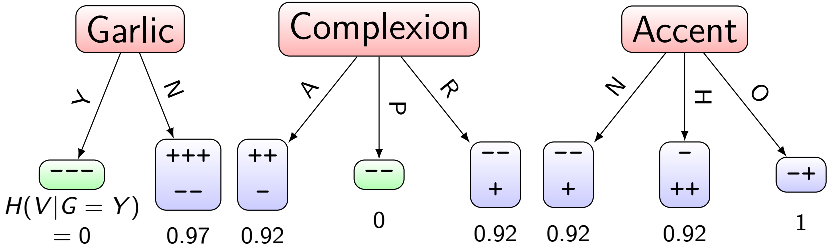

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$H(V|S)$ is maximum. Thus should pick test S first

Potential extensions

- The test does not need to return discrete result. Let X be the test outcome. It can be continuous as well

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- Build a number of trees instead of a single tree $\Rightarrow$ random forests

Random forests

- Pick random subset of training samples
- Train on each random subset but limited to a subset of features/attributes
- Given a test sample
    - Classify sample using each of the trees
    - Make final decision based on majority vote

Law of Large Number (LLN)

If we randomly sample $x_1, x_2, \cdots, x_N$ from an i.i.d. (identical and independently distributed) source, the average of $f(x_i)$ will approach the expected value as $N \to \infty$. That is,

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Example

This is precisely how poll supposes to work! Pollster randomly draws sample from a portion of the population but will expect the prediction matches the outcome

Proof of LLN

The LLN is a rather strong result. We will only show a weak version here

$$Pr\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(X_i) - E[f(X)]\right| \geq a\right) \leq \frac{Var(f(X))}{Na^2} \propto \frac{1}{N}$$

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Main idea

Consider a sequence of symbols $x_1, x_2, \cdots, x_N$ sampled from a DMS and consider the sample average of the log-probabilities of each sampled symbols

$$\frac{1}{N}\sum_{i=1}^{N}\log\frac{1}{p(x_i)} \to E\left[\log\frac{1}{p(X)}\right]$$

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Rearranging the terms, this implies that for any sequence sampled from the source, the probability of the sampled sequence $p(x^N) \to 2^{-NH(X)}$!

Set of typical sequences

Let's name the sequence x^N with $p(x^N) \sim 2^{-NH(X)}$ typical and define the set of typical sequences

$$\mathcal{A}_\epsilon^N(X) = \{x^N | 2^{-N(H(X)+\epsilon)} \leq p(x^N) \leq 2^{-N(H(X)-\epsilon)}\}$$

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- For any $\epsilon > 0$, we can find a sufficiently large N such that any sampled sequence from the source is typical
- Since all typical sequences have probability $\sim 2^{-NH(X)}$ and they fill up the entire probability space (everything is typical), there should be approximately $\frac{1}{2^{-NH(X)}} = 2^{NH(X)}$ typical sequences

Precise bounds on the size of typical set

$$(1-\delta)2^{N(H(X)-\epsilon)} \leq |\mathcal{A}_\epsilon^N(X)| \leq 2^{N(H(X)+\epsilon)}$$

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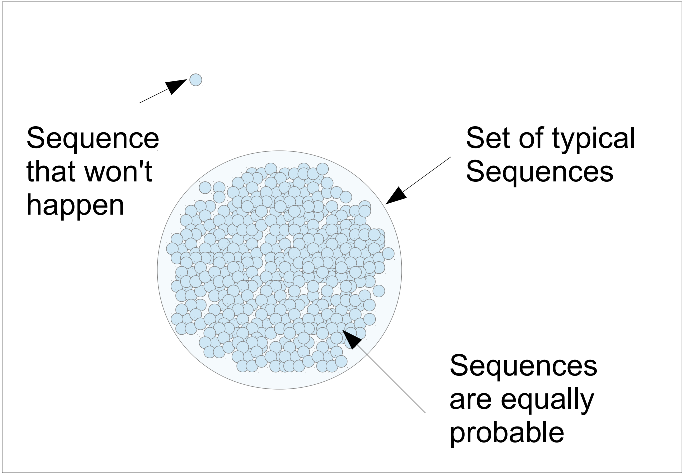
AEP



Asymptotic equipatition refers to the fact that the probability space is equally partitioned by the typical sequences

AEP and compression limit

Consider coin flipping again, let say $Pr(Head) = 0.3$ and $N = 1000$

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- AEP also tells us that the number of typical sequences are approximately $2^{NH(X)}$
- Therefore, we can simply assign index to all the typical sequences and ignore the rest. Then we only need $\log 2^{NH(X)} = NH(X)$ to store a sequence of N symbols. And on average, we need $H(X)$ bits per symbol as before!

Previously...

- Identification/Decision trees
- Random forests
- Law of Large Number
- Asymptotic equipartition (AEP) and typical sequences

This time

- Joint typical sequences
- Covering and Packing Lemmas
- Channel coding setup
- Channel coding rate
- Channel capacity
- Channel Coding Theorem

Jointly typical sequences

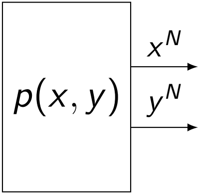
For a pair of sequences x^N and y^N, we say that they are jointly typical if

$$2^{-N(H(X,Y)+\epsilon)} \leq p(x^N, y^N) \leq 2^{-N(H(X,Y)-\epsilon)}$$

and x^N and y^N themselves are typical

As in the single sequence case,

- Any sequence pair drawing from a joint source $p(x,y)$ is essentially jointly typical
- There are $\sim 2^{NH(X,Y)}$ jointly typical sequences



Joint typicality of independent seqences

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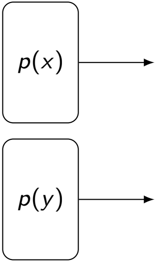

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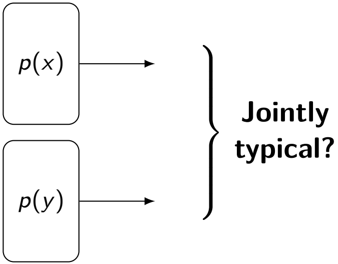

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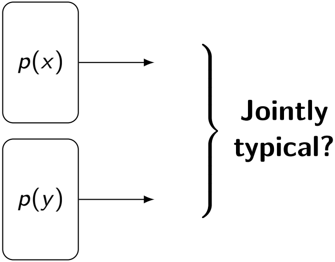

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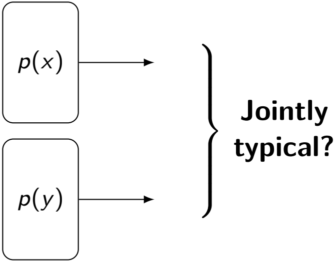

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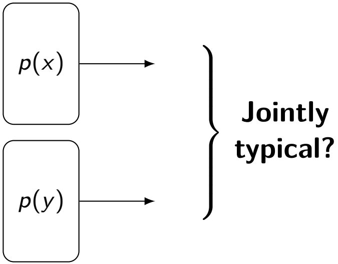

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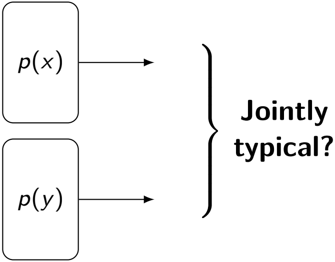

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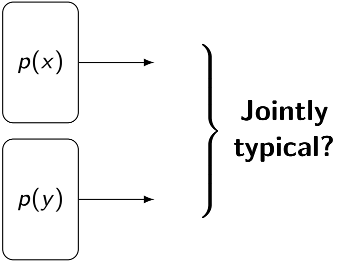

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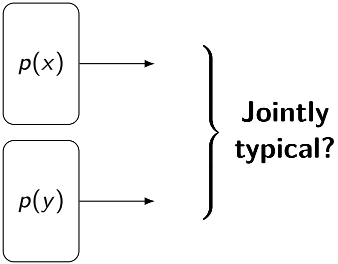
&\geq \sum_{\{(x^N,y^N)|(x^N,y^N)\in\mathcal{A}_\epsilon^{(N)}\}} 2^{-N(H(X)+\epsilon)}2^{-N(H(Y)+\epsilon)}
\end{aligned}
$$



Joint typicality of independent seqences

- Given sequences X^N and Y^N independently drawn from discrete memoryless sources $p(x)$ and $p(y)$
- What is the probability that X^N and Y^N are jointly typical?

$$
\begin{aligned}
&Pr((X^N,Y^N)\in\mathcal{A}_\epsilon^{(N)}) \\
&= \sum_{\{(x^N,y^N)|(x^N,y^N)\in\mathcal{A}_\epsilon^{(N)}\}} p(x^N,y^N) \\
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&\geq (1-\delta)2^{-N(I(X;Y)+3\epsilon)}
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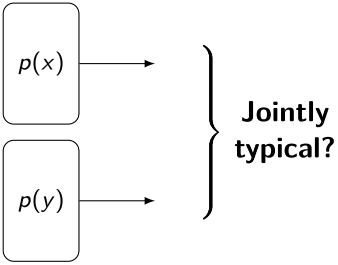
$$



Packing lemma

How many independent Y^N sequences can pack with some X^N without becoming jointly typical with X^N?

- Say, M Y^N sequences were drawn

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\begin{aligned}
&Pr(\text{Any one of } M \ Y^N \text{ jointly typical with } X^N) \\
\leq & M Pr((X^N, Y^N) \in \mathcal{A}_\epsilon^N(X, Y)) \\
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\leq &2^{-N(I(X;Y)-R-3\epsilon)} \rightarrow 0 \text{ as } N \rightarrow \infty \text{ and } I(X;Y) - 3\epsilon > R,
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\end{aligned}$$

where $2^{NR} = M$

Since ϵ can be made arbitrarily small as N increases, as long as $I(X;Y) > R$, we can find a sufficiently large N so that we can "pack" the M Y^N with X^N and none of the Y^N will be jointly typical with X^N

Covering lemma

How many independent Y^N are needed until it is jointly typical with X^N?

- Again, draw $M(=2^{NR})$ Y^N sequences
- Under what condition that *at least one* Y^N jointly typical with X^N?

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=& \prod_{m=1}^{M} \left[1 - Pr((X^N(m), Y^N) \in \mathcal{A}_\epsilon^{(N)}(Y, X))\right] \\
\leq& (1-(1-\delta)2^{-N(I(Y;X)+3\epsilon)})^M
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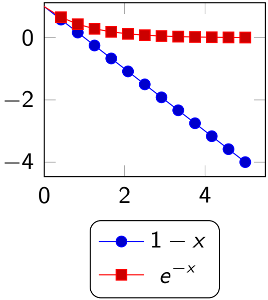

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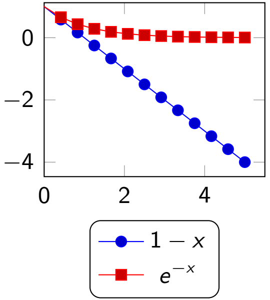


Summary of packing lemma and covering lemma

Packing Lemma

We can "pack" $M = 2^{NR}$ (with $R < I(X;Y)$) x^N together without being jointly typical with y^N

Covering Lemma

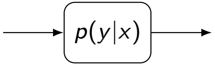
We can "cover" with $M = 2^{NR}$ (with $R > I(X;Y)$) x^N such that at least one x^N being jointly typical with y^N

Remark

- Packing lemma is useful in the proof of channel coding theorem
- Covering lemma is useful in the proof of rate-distortion theorem

We will look into the above applications later in this course

Channel coding setup



- As the name suggests, the output of a discrete memoryless channel (DMS) only depends on the current input (thus no memoryless). And both its input X and output Y are characterized by the conditional probability $p(y|x)$

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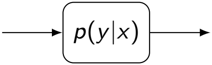

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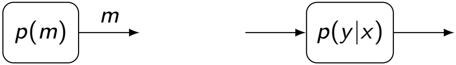

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Channel coding setup

$$p(m) \xrightarrow{m} \textit{Encoder} \longrightarrow p(y|x) \longrightarrow \textit{Decoder}$$

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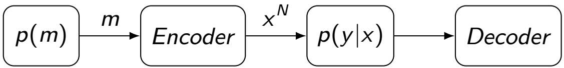

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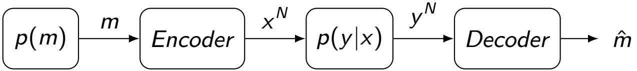

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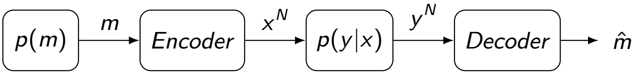
- Given a message m (say generated from a distribution $p(m)$)
    - We will have an encoder decoder pair
    - The encoder will convert m to x^N suitable for transmission
    - Decoder will try to extracted the message from the channel output y^N

Channel coding rate

$$\boxed{p(m)} \xrightarrow{m} \boxed{Encoder} \xrightarrow{x^N} \boxed{p(y|x)} \xrightarrow{y^N} \boxed{Decoder} \longrightarrow \hat{m}$$

The channel coding rate is defined as number of bits of message can be sent per channel use

Channel coding rate



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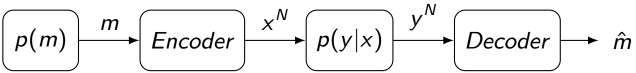

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- $R = \frac{H(M)}{N}$

Channel capacity

- By Shannon's channel coding theorem, the capacity of the channel (will be shown later) is given by

$$C = \max_{p(x)} I(X; Y)$$

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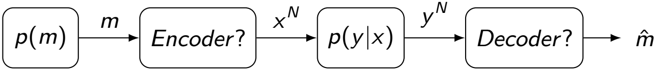
- On the other hand, if R is larger than the capacity C, no matter how we try, it is impossible to recontruct m error free
- An intuitive interpretation is that the amount of information can be passed through a channel is just mutual information between the input and output. And since we can pick the statistics of our input, we may make our choice wisely and maximize the mutual information. And the maximum that we can attain is the capacity

Continuous channel



Continuous channel

$$p(m) \xrightarrow{m} \textit{Encoder} \xrightarrow{x_\Delta^N} D/A \xrightarrow{x^N} p(y|x) \xrightarrow{y^N} A/D \xrightarrow{y_\Delta^N} \textit{Decoder} \longrightarrow \hat{m}$$

- For continuous channel, we can create a "pseudo" discrete channel using A/D and D/A converters

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- For continuous channel, we can create a "pseudo" discrete channel using A/D and D/A converters
- The maximum information that can pass through the channel will then be

$$\begin{aligned} C_\Delta &= \max_{p(x)} I(X_\Delta; Y_\Delta) = \max_{p(x)} H(Y_\Delta) - H(Y_\Delta|X_\Delta) \\ &\approx \max_{p(x)} h(Y) - \log\Delta - h(Y|X_\Delta) + \log\Delta \\ &\approx \max_{p(x)} h(Y) - h(Y|X) = \max_{p(x)} I(X;Y) \end{aligned}$$

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- As $\Delta \to 0$, $C = \max_{p(x)} I(X;Y)$. So expression is completely the same as the discrete case

Example: Binary symmetric channel

- Both input and output are binary (say take value 0 or 1)

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&= \max_{p(x)} h(Y) - \frac{1}{2}\log 2\pi e \sigma_Z^2
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&= \max_{p(x)} h(Y) - \frac{1}{2}\log 2\pi e\sigma_Z^2 = \frac{1}{2}\log 2\pi e\sigma_Y^2 - \frac{1}{2}\log 2\pi e\sigma_Z^2
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&= \max_{p(x)} h(Y) - h(Y|X) = \max_{p(x)} h(Y) - h(X+Z|X) \\
&= \max_{p(x)} h(Y) - h(Z|X) = \max_{p(x)} h(Y) - h(Z) \\
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&= \frac{1}{2}\log\frac{\sigma_X^2+\sigma_Z^2}{\sigma_Z^2} = \frac{1}{2}\log\left(1+\frac{\sigma_X^2}{\sigma_Z^2}\right)
\end{aligned}$$

Example: Gaussian channel

The channel output $Y = X + Z$, where Z is a zero-mean Gaussian noise (independent of the input X)

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where SNR is the signal to noise ratio

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Consider an AWGN channel with bandwidth W and two-sided power spectrum density of $N_0/2$

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Codebook construction

Forward statement

If the code rate $R < C = \max_{p(x)} I(X;Y)$, according to the Channel Coding Theorem, we should be able to find a code with encoding mapping $\mathbf{c} : m \in \{1, 2, \cdots, 2^{NR}\} \to \{0, 1\}^N$ and the error probability of transmitting any message $m \in \{1, 2, \cdots, 2^{NR}\}$, $p_e(m)$, is arbitrarily small

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- The main tool of the proof is random coding
- Let $p^*(x) = \arg\max_{p(x)} I(X;Y)$. Generate codewords from the DMS $p^*(x)$ by sampling 2^n length-n sequences from the source:

$$
\begin{aligned}
\mathbf{c}(1) =& (x_1(1), x_2(1), \cdots, x_N(1)) \\
\mathbf{c}(2) =& (x_1(2), x_2(2), \cdots, x_N(2)) \\
& \cdots \\
\mathbf{c}(2^{NR}) =& (x_1(2^{NR}), x_2(2^{NR}), \cdots, x_N(2^{NR}))
\end{aligned}
$$

Encoding and decoding

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Decoding

Upon receiving sequence $\mathbf{y} = (y_1, y_2, \cdots, y_N)$, pick the sequence $\mathbf{c}(m)$ from $\{\mathbf{c}(1), \cdots, \mathbf{c}(2^{NR})\}$ such that $(\mathbf{c}(m), \mathbf{y})$ are jointly typical. That is $p_{X^N,Y^N}(\mathbf{c}(m), \mathbf{y}) \sim 2^{-nH(X,Y)}$. If no such $\mathbf{c}(m)$ exists or more than one such sequence exist, announce error. Otherwise output the decoded message as m

Average performance

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Since ϵ can be made arbitrarily small as N increase, as long as $I(X; Y) - 3\epsilon > R$, we can make P_2 arbitrarily small also given a sufficiently large N

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- If we discard worse half of the codewords, for remaining m, we have $Pr(error|\mathbf{c}^*, m) \leq 2Pr(error|\mathbf{c}^*) = 2\delta \to 0$ as $N \to \infty$
- Even though the rate reduces from R to $R - \frac{1}{N}$ (number of messages from $2^{NR} \to 2^{NR-1}$). But we can still make the final rate arbitrarily close to the capacity as $N \to \infty$

Previously...

- Joint typical sequences
- Covering and Packing Lemmas
- Channel Coding Theorem
- Capacity of Gaussian channel
- Capacity of additive white Gaussian channel
- Forward proof of Channel Coding Theorem

This time

- Converse Proof of Channel Coding Theorem
- Non-white Gaussian Channel
- Rate-distortion problems
- Rate-distortion Theorem

Converse proof

We want to say that whenever the code rate is larger than the capacity, the probability of error will be non-zero

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To continue the converse proof, we will need to introduce a simple result from Fano

Fano's inequality

Fano's inequality

Denote $Pr(error) = P_e = Pr(M \neq \hat{M})$, then $H(M|Y^N) \leq 1 + P_e H(M)$
Intuitively, if $P_e \to 0$, on average we will know M for certain given y and thus $\frac{1}{N} H(M|Y^N) \to 0$

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$$\begin{aligned} H(M|Y^N) &= H(M,E|Y^N) - H(E|Y^N, M) \\ &= H(M,E|Y^N) = H(E|Y^N) + H(M|Y^N, E) \end{aligned}$$

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&\leq H(E) + H(M|Y^N, E) \\
&\leq 1 + P(E=0)H(M|Y^N, E=0) + P(E=1)H(M|Y^N, E=1) \\
&\leq 1 + 0 + P_e H(M|Y^N, E=1) \overset{(d)}{\leq} 1 + P_e H(M)
\end{aligned}
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$$R = \frac{H(M)}{N} = \frac{1}{N}\left[I(M;Y^N) + H(M|Y^N)\right]$$

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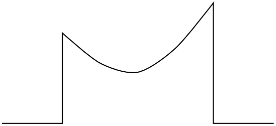
as $N \to \infty$ by Fano's inequality

Color channels

- We look into capacity of white Gaussian channel last time

Color channels



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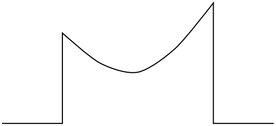

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- Intuitively, we should assign different amount of power to different band. Hence, we have an allocation problem
- Without loss of generality, let's consider the discrete approximation, parallel Gaussian channel

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- Consider that we have K parallel channels (K bands) and the corresponding noise powers are $\sigma_1^2, \sigma_2^2, \cdots, \sigma_K^2$

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Parallel Gaussian channels

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- So our goal is to assign $P_1, P_2, \cdots, P_K \geq 0$ ($\sum_{k=1}^K P_k \leq P$) such that the total capacity

$$\sum_{k=1}^{K} \frac{1}{2}\log\left(1+\frac{P_k}{\sigma_k^2}\right)$$

is maximize

KKT conditions

Let's list all the KKT conditions for the optimization problem

$$\max \sum_{k=1}^{K} \frac{1}{2} \log \left(1 + \frac{P_k}{\sigma_k^2}\right) \qquad \text{such that}$$

$$P_1, \cdots, P_K \geq 0, \qquad \sum_{k=1}^{K} P_k \leq P$$

$$\frac{\partial}{\partial P_i} \left[\sum_{k=1}^{K} \frac{1}{2} \log \left(1 + \frac{P_k}{\sigma_k^2}\right) + \sum_{k=1}^{K} \lambda_k P_k - \mu \left(\sum_{k=1}^{K} P_k - P \right) \right] = 0$$

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$$\mu\left(\sum_{k=1}^{K} P_k - P\right) = 0, \qquad \lambda_k P_k = 0, \forall k$$

Capacity of parallel channels

$$\frac{\partial}{\partial P_i}\left[\sum_{k=1}^{K}\frac{1}{2}\log\left(1+\frac{P_k}{\sigma_k^2}\right)+\sum_{k=1}^{K}\lambda_k P_k-\mu\left(\sum_{k=1}^{K}P_k-P\right)\right]=0$$

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Since $\lambda_i P_i=0$, for $P_i>0$, we have $\lambda_i=0$ and thus

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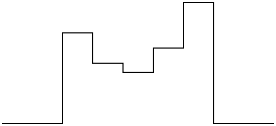
Since $\lambda_i P_i = 0$, for $P_i > 0$, we have $\lambda_i = 0$ and thus

$$P_i+\sigma_i^2=\frac{1}{2\mu}=constant$$

This suggests that $\mu > 0$ and thus $\sum_{k=1}^{K} P_k = P$

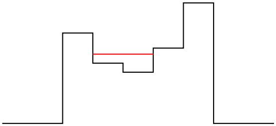
Water-filling interpretation



From $P_i + \sigma_i^2 = const$, power can be allocated intuitively as filling water to a pond (hence "water-filling")

Example

Water-filling interpretation



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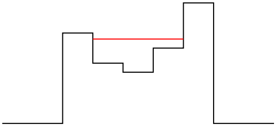

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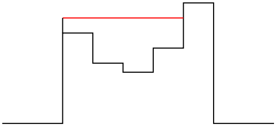

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