Information Theory and Probabilistic Programming

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2. Statistical inference
    - Bayesian and Monte Carlo techniques
3. Introduction of probabilistic programming
    - Solve inference problems with programming

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- Study of “information” using probability
- Can be treated as a subfield of applied probability
- But it has a huge impact to communications and information science
  - The theoretical basis of the entire telecom industry is built on top of that
  - Study of extreme cases. What is possible and what is not?

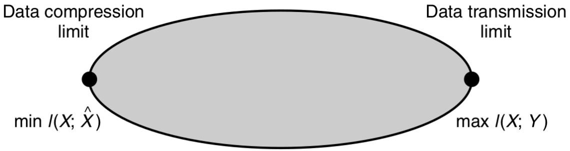

FIGURE 1.2. Information theory as the extreme points of communication theory.

(From Cover and Thomas)

Connection to other fields

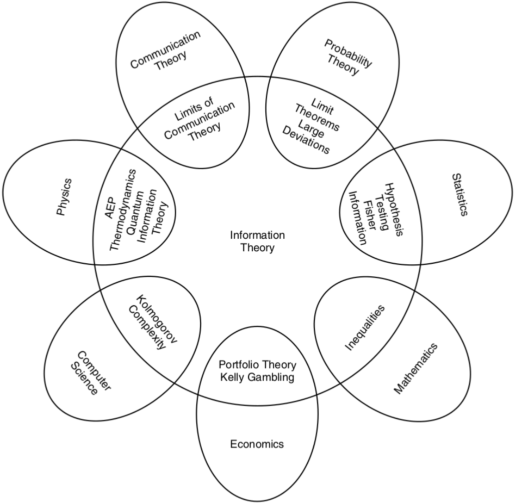
Communication Theory
Probability Theory
Limits of Communication Theory
Limit Theorems Large Deviations
Physics
AEP Thermodynamics Quantum Information Theory
Information Theory
Hypothesis Testing Fisher Information
Statistics
Kolmogorov Complexity
Computer Science
Inequalities
Mathematics
Portfolio Theory Kelly Gambling
Economics

FIGURE 1.1. Relationship of information theory to other fields.

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        - Give the capacity of Gaussian channel as an example

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- Some similar ideas were explored earlier in Bell Labs by Harry Nyquist and Ralph Hartley. But those results are limited to events with equal probability

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- A Preview:

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A good guess for $H(X=x)$: $\log \frac{1}{p(x)}$

Computer scientists' treatment

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- We will take the probabilistic view (electrical/communication engineers treatment here) to quantify information theory who usually study with Bayesian models

Neumann-Shannon Anecdote

When Shannon discovered this function he was faced with the need to name it, for it occurred quite often in the theory of communication he was developing. He considered naming it “information” but felt that this word had unfortunate popular interpretations that would interfere with his intended uses of it in the new theory. He was inclined towards naming it “uncertainty” and discussed the matter with the late John Von Neumann. Von Neumann suggested that the function ought to be called “entropy” since it was already in use in some treatises on statistical thermodynamics (e.g. ref. 12). Von Neumann, Shannon reports, suggested that there were two good reasons for calling the function “entropy”. ”It is already in use under that name,” he is reported to have said, ”and besides, it will give you a great edge in debates because nobody really knows what entropy is anyway.” Shannon called the function “entropy” and used it as a measure of “uncertainty,” interchanging the two words in his writings without discrimination.

–From wikipedia

Some probability basic

- Probability mass function (pmf) for discrete random variable (r.v.) X
  - $p(x) \geq 0$
  - $p(x) \leq 1$
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- Marginalization: $\sum_x p(x, y) = p(y)$
- Conditional probability (Bayes' rule): $p(x|y) = \frac{p(x,y)}{p(y)}$
  - N.B. $\sum_x p(x|y) = 1$ but $\sum_y p(x|y) \neq 1$

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- Chain rule: $p(x, y, z) = p(x)p(y|x)p(z|x, y)$
  $RHS = p(x)p(y|x)p(z|x, y) = p(x)\frac{p(x,y)}{p(x)}\frac{p(x,y,z)}{p(x,y)} = p(x, y, z) = LHS$

- Independence: $p(x,y)=p(x)p(y)$, $X \perp\!\!\!\perp Y$
  - By Bayes/chain rule, $p(x,y)=p(x)p(y|x)$. Therefore the condition implies that $p(y|x)=p(y)$. In other words, no matter what value X takes, the probability of Y given X is not going to change. So reasonably, they are independent

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- Markov property and conditional independence: $p(x,y|z)=p(x|z)p(y|z)$, $X \perp\!\!\!\perp Y|Z, X \leftrightarrow Z \leftrightarrow Y$
  - Similar to independence, by chain rule, we have $p(x,y|z)=p(x|z)p(y|x,z)$. Along with the above condition, $p(y|x,z)=p(y|z)$. Thus given Z, it does not matter what X supposed to be, the probability of given both variables will not depend on X. Hence, X and Y are conditionally independent given Z
- Caveat: independence and conditional independence are two "independent concepts", we can have both satisfied, none of them satisfied, or one of them satisfied. A common **mistake** is to think that independence leads to conditional independence or vice versa. But that is WRONG

Independence but not conditional independence

Consider flipping two coins with outcomes store as X and Y, say 1 represents a head and 0 represents a tail

- In general the two outcomes should be independent (maybe unless if you are some professional/magical gambler), so we have $X \perp\!\!\!\perp Y$
- Now, let $Z = X \oplus Y$, where $\oplus$ is the exclusive or operation ($1 \oplus 0 = 0 \oplus 1 = 1$ and $1 \oplus 1 = 0 \oplus 0 = 0$)
    - Even though $X \perp\!\!\!\perp Y$, $X \not\perp\!\!\!\perp Y|Z$
    - Actually given Z, X "depends" very much on Y since from $X = Y \oplus Z$, we can find out X precisely given Y
    - We can also check the condition $X \perp\!\!\!\perp Y|Z$ by comparing the probability $p(x|z,y)$ with $p(x|z)$
        - For example, $p_{X|Z}(0|0) = 0.5 \neq 1 = p_{X|Z,Y}(0|0,0)$. Thus $X \perp\!\!\!\perp Y|Z$ cannot be true

More formal treatment: probability space

- More rigorously, a probability model is defined by the **probability space** composed of the triple $(\Omega, \mathcal{F}, p)$
  - Ω is the **sample space** containing all possible outcomes
  - $\mathcal{F}$ is a "σ-field", which is a collection of subsets (events) of Ω
  - p is the (non-negative) **probability measure** on elements of $\mathcal{F}$
- E.g., probability model of unbiased dice
  - $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \cdots, \{1, 2, 3, 4, 5, 6\}\}$
  - $p(S)$ is the probability of an event
    - $p(\{1\}) = p(\{2\}) = p(\{3\}) = p(\{4\}) = p(\{5\}) = p(\{6\}) = 1/6$
    - $p(\{1, 2\}) = p(\{1, 3\}) = \cdots = p(\{5, 6\}) = 2/6$
    - $\cdots$
    - $p(\{1, 2, 3, 4, 5, 6\}) = 1$
- N.B. It could be confusing at first. Be careful that events $\neq$ outcomes. **An event is actually a set of outcomes**

σ-algebra

- The purpose of σ-field (aka σ-algebra) is to impose restriction on what we can and cannot query regarding probability
- Namely, we can only measure the probability of something inside the σ-field $\mathcal{F}$ (i.e., an event)
- Formal definition of σ-field:
  - **σ-field has to satisfied the following: 1) containing empty set $\varnothing$, 2) closed under complement, countable union, and countable intersection of its element**
- E.g., let $\Omega = \{1, 2, 3, 4\}$
  1. $\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is a valid σ-field
  2. $\{\varnothing, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is NOT a valid σ-field
- N.B., A complement, countable union, or countable intersection of Ω is call a **Borel set**
  - $\varnothing, \{1\}, \{1, 2\}$ are example of Borel sets (an event is a Borel set)
  - Collection of all Borel sets forms a σ-algebra (aka Borel (σ-)algebra)

Probability measure

- Probability measure p is a **measure**. Along with $\mathcal{F}$, the tuple $(\mathcal{F}, p)$ forms a **measure space**. For $\mathbb{P}$ to be a valid probability measure, it has to satisfy the following
  - Requirements to be a measure (in the context of measure theory):
    1. $p(\varnothing) = 0$
    2. Countably additive: $p(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} p(A_i), \forall i \neq j, A_i \cap A_j = \varnothing$
  - And since p is a probability measure, it also has to satisfy $p(\Omega) = 1$
- The above constraints are sometimes known as the axioms of probability theory

Some properties of probability measure

From the axioms described in the last slides, one can show that probability measure has to satisfies the following:

1. $p(A^c) = 1 - p(A)$
2. $p(A) \leq p(B)$ if $A \subset B$
3. Union bound: $p(\cup_i A_i) \leq \sum_i p(A_i)$
    - Proof hint: use 2) and induction
4. Inclusion-exclusion formula: $p(\cup_{i=1}^n A_i) = \sum_{i=1}^n p(A_i) - \sum_{i<j} p(A_i \cap A_j) + \sum_{i<j<k} p(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} p(\cap_{i=1}^n A_i)$
    - Proof hint: show $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ and then use induction. ($p(A \cup B) = p(A) + p(B \setminus A)$ and $p(B) = p(A \cap B) + p(B \setminus A)$).

Why so complex?

- Consider X a uniform random variable defined between $[0,1]$
- Define $Y = \begin{cases} 1 & \text{if } X \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$
- Y is a random variable since X is random. It is reasonable to ask what is the probability that $Y=1$. From undergrad probability class,

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dx = ?$$

  - The integral above is actually undefined according to undergrad calculus, where the integral is known as a Riemann integral
- Instead, we have to incorporate the idea of "measure" (Lesbeque integral)

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dp(x) = 0$$

  - The Lesbeque integral above is 0 since the measure of $\{x|x\in[0,1]\cap\mathbb{Q}\} = 0$

Some remarks on notation

- In general, we can write

$$p(\Omega') = \int_{\Omega'} dp(\omega)$$

and

$$E[f(X)] = \int_{\Omega} f(X(\omega))dp(\omega)$$

- E.g.,

$$E[X] = \int_{\Omega} X(\omega)dp(\omega) = \int_{\Omega} X(\omega)\ dp = \int_{\Omega} Xdp$$

  - Note that p is the probability measure (often people use upper case P instead)
  - People often omit ω as above when context is clear

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$

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  $p(\theta, o|M) = p(o|M)p(\theta|o, M) = p(\theta|M)p(o|\theta, M)$

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$$\underbrace{p(\theta|o, M)}_{posterior} = \frac{\overbrace{p(\theta|M)}^{prior}\overbrace{p(o|\theta, M)}^{likelihood}}{\underbrace{p(o|M)}_{model\ \ evidence}}$$

- M: model type
- θ: model parameter
- o: observation

Inference

o: Observed variable, θ: Parameter, x: Latent variable

Maximum Likelihood (ML)

$$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(o|\theta)$$

Maximum A Posteriori (MAP)

$$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(\theta|o)$$

Bayesian

$$\hat{x} = \textstyle\sum_x x \underbrace{\sum_\theta p(x|\theta)p(\theta|o)}_{p(x|o)}$$

where $p(\theta|o) = \frac{p(o|\theta)p(\theta)}{p(o)} \propto p(o|\theta)\underbrace{p(\theta)}_{prior}$

Coin Flip

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

Which coin will I use?

$P(C_1) = 1/3$ $P(C_2) = 1/3$ $P(C_3) = 1/3$

Prior: Probability of a hypothesis
before we make any observations

(Slide credit: University of Washington CSE473)

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Which coin will I use?

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Uniform Prior: All hypothesis are equally likely before we make any observations

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = ?$ $P(C_2|H) = ?$ $P(C_3|H) = ?$

$$P(C_1|H) = \frac{\boxed{P(H|C_1)}\boxed{P(C_1)}}{\boxed{P(H)}} \qquad \boxed{P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)}$$

C_1

$\boxed{P(H|C_1) = 0.1}$

$\boxed{P(C_1) = 1/3}$

C_2

$P(H|C_2) = 0.5$

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C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

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Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.6$

Posterior: Probability of a hypothesis given data

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = ?$ $P(C_2|HT) = ?$ $P(C_3|HT) = ?$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.21$ $P(C_2|HT) = 0.58$ $P(C_3|HT) = 0.21$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

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C_3

$P(H|C_3) = 0.9$

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$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:

C_2

Best estimate for P(H)

$P(H|C_2) = 0.5$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum Likelihood Estimate: The best hypothesis that fits observed data assuming uniform prior

Most likely coin:

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Best estimate for P(H)

$P(H|C_2) = 0.5$

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$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

- Should we always use Uniform Prior?
- Background knowledge:
  - Heads => you go first in Abalone against TA
  - TAs are nice people
  - => TA is more likely to use a coin biased in your favor

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

We can encode it in the prior:

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

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Which coin did I use?

$P(C_1|H)$ = ? $P(C_2|H)$ = ? $P(C_3|H)$ = ?

$$P(C_1|H) = \alpha P(H|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = 0.006$ $P(C_2|H) = 0.165$ $P(C_3|H) = 0.829$

ML posterior after Exp 1:

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.600$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 0.05$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 0.25$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = ?$ $P(C_2|HT) = ?$ $P(C_3|HT) = ?$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 0.05$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 0.25$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:

C_3

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum A Posteriori (MAP) Estimate: The best hypothesis that fits observed data assuming a non-uniform prior

Most likely coin:

C_3

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_2 and C_3 are almost equally likely

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

A Better Estimate

Recall: $P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)$ = 0.680

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Bayesian Estimate

Bayesian Estimate: Minimizes prediction error, given data and (generally) assuming a non-uniform prior

$$P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i) = 0.680$$

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Comparison

ML

- Easy to compute

Comparison

ML
- Easy to compute

MAP
- Still relatively easy to compute
- Incorporate prior information

Comparison

ML

- Easy to compute

MAP

- Still relatively easy to compute
- Incorporate prior information

Bayesian

- Minimizes expected error ⇒ especially shines when little data available
- Potentially much harder to compute