Information Theory and Probabilistic Programming

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1. Lecture 1
   - Introduction
   - Review of probabilities
   - Introduction to Monte Carlo
   - Appendix

2. Lecture 2
   - Introduction to probabilistic inference

Lecture 1

About this course

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3. Introduction of probabilistic programming
   - Solve inference problems with programming
4. Get better understanding of probability

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- But it has a huge impact to communications and information science
  - The theoretical basis of the entire telecom industry is built on top of that
  - Study of extreme cases. What is possible and what is not?

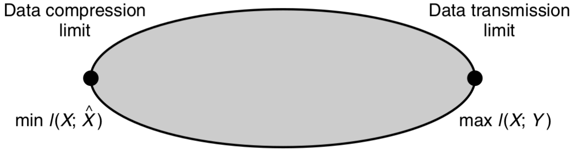

FIGURE 1.2. Information theory as the extreme points of communication theory.

(From Cover and Thomas)

Connection to other fields

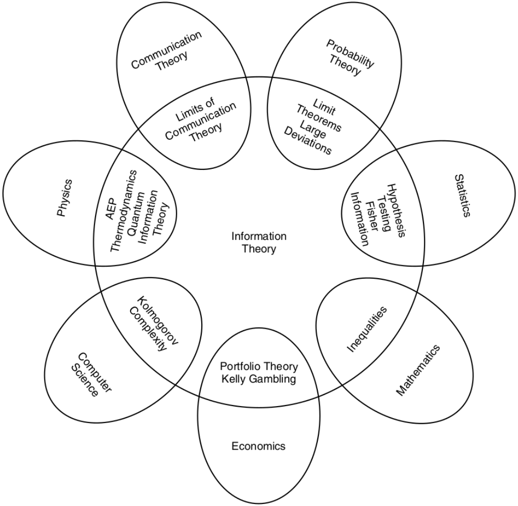

FIGURE 1.1. Relationship of information theory to other fields.

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- Some similar ideas were explored earlier in Bell Labs by Harry Nyquist and Ralph Hartley. But those results are limited to events with equal probability

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A good guess for $H(X=x)$: $\log \frac{1}{p(x)}$

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- We will take the probabilistic view (electrical/communication engineers treatment here) to quantify information theory who usually study with Bayesian models

Neumann-Shannon Anecdote

When Shannon discovered this function he was faced with the need to name it, for it occurred quite often in the theory of communication he was developing. He considered naming it “information” but felt that this word had unfortunate popular interpretations that would interfere with his intended uses of it in the new theory. He was inclined towards naming it “uncertainty” and discussed the matter with the late John Von Neumann. Von Neumann suggested that the function ought to be called “entropy” since it was already in use in some treatises on statistical thermodynamics (e.g. ref. 12). Von Neumann, Shannon reports, suggested that there were two good reasons for calling the function “entropy”. ”It is already in use under that name,” he is reported to have said, ”and besides, it will give you a great edge in debates because nobody really knows what entropy is anyway.” Shannon called the function “entropy” and used it as a measure of “uncertainty,” interchanging the two words in his writings without discrimination.
–From wikipedia

Probability model

- A probability model is used to model uncertain event that can have non-deterministic outcomes
- A probability model can have finite or infinite number of outcomes and even continuous outcomes
- We call the "undetermine" random variable, short for r.v.
- The probability of an **outcome** is the relative chance of getting that outcome
  - For outcome a, we may denote as $Pr(X = a)$ or $p_X(a)$ or even $p(a)$ when it is understood that we are considering variable X
  - $0 \leq p(a) \leq 1$
- We often denote a r.v. using upper case (such as X) and its realization (what was actually observed) using lower case (such as x)

Some probability properties

- Probability mass function (pmf) for discrete random variable (r.v.) X
  - $p(x) \geq 0$
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- Chain rule: $p(x,y,z) = p(x)p(y|x)p(z|x,y)$
  $RHS = p(x)p(y|x)p(z|x,y) = p(x)\frac{p(x,y)}{p(x)}\frac{p(x,y,z)}{p(x,y)} = p(x,y,z) = LHS$

Probabilities and counting

- Six students A, B, C, D, E, F randomly lined up in a row, what is the probability that the order is exactly ABCDEF?
- Six students randomly assigned into two teams (black and white), what is the probability that A,B,C assigned to Team Black and the rest assigned to Team White?

Example: Two jars

- Both Jars A and B have 4 balls
    - Jar A has 1 white and 3 black
    - Jar B has 2 white and 2 black
- Let's draw balls from the jars multiple times. **And put the drawn ball back after each draw.** Can you answer the following?
    - What is the probability of get a white ball from Jar A?
    - What is the probability of getting 3 whites after 6 drawings?
    - If someone randomly pick a jar to draw from and get 3 whites after 6 drawing, what is the probability that he drew from Jar A?

Bayes rule

- Both Jars A and B have 4 balls
  - Jar A has 1 white and 3 black
  - Jar B has 2 white and 2 black
- Say probability of picking Jar A, $Pr(Jar = A) = 0.5$
  - What is the probability of picking from Jar A and getting a white ball $Pr(Jar = A, Ball = white)$?
  - What is $Pr(Ball = white|Jar = A)$?
  - What is $Pr(Jar = A|Ball = white)$?

Expectation

- Recall that $p(x)$ as the distribution of a r.v. X
- The expected value of X is $E[X] \triangleq \sum_x x \cdot p(x)$
- In general, the expected value of a function $f(\cdot)$ of X is $E[f(X)] \triangleq \sum_x f(x) \cdot p(x)$
- Examples
  - $E[X]$ is just the mean of X, often denote as $\overline{X}$
  - The variance of X is $E[(X - \overline{X})^2]$

Independence and conditional independence

- Independence: $p(x, y) = p(x)p(y)$, $X \perp\!\!\!\perp Y$
  - By Bayes/chain rule, $p(x, y) = p(x)p(y|x)$. Therefore the condition implies that $p(y|x) = p(y)$. In other words, no matter what value X takes, the probability of Y given X is not going to change. So reasonably, they are independent

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- Markov property and conditional independence: $p(x, y|z) = p(x|z)p(y|z)$, $X \perp\!\!\!\perp Y|Z, X \leftrightarrow Z \leftrightarrow Y$
    - Similar to independence, by chain rule, we have $p(x, y|z) = p(x|z)p(y|x, z)$. Along with the above condition, $p(y|x, z) = p(y|z)$. Thus given Z, it does not matter what X supposed to be, the probability of given both variables will not depend on X. Hence, X and Y are conditionally independent given Z
- Caveat: independence and conditional independence are two "independent concepts", we can have both satisfied, none of them satisfied, or one of them satisfied. A common **mistake** is to think that independence leads to conditional independence or vice versa. But that is WRONG

Independence but not conditional independence

Consider flipping two coins with outcomes store as X and Y, say 1 represents a head and 0 represents a tail

- In general the two outcomes should be independent (maybe unless if you are some professional/magical gambler), so we have $X \perp\!\!\!\perp Y$
- Now, let $Z = X \oplus Y$, where $\oplus$ is the exclusive or operation $(1 \oplus 0 = 0 \oplus 1 = 1$ and $1 \oplus 1 = 0 \oplus 0 = 0)$
  - Even though $X \perp\!\!\!\perp Y$, $X \not\perp\!\!\!\perp Y|Z$
  - Actually given Z, X "depends" very much on Y since from $X = Y \oplus Z$, we can find out X precisely given Y
  - We can also check the condition $X \perp\!\!\!\perp Y|Z$ by comparing the probability $p(x|z,y)$ with $p(x|z)$
    - For example, $p_{X|Z}(0|0) = 0.5 \neq 1 = p_{X|Z,Y}(0|0,0)$. Thus $X \perp\!\!\!\perp Y|Z$ cannot be true

A digression: Naive Bayes Algorithm

- Naive Bayes is a simple machine learning algorithm to classify an object with its features
- Basically, we are simply assuming the features are conditionally independent given the object class
- Say if O is the object that $c(O)$ is the corresponding class (can be $c_1, c_2, \cdots$). And say $f_1(O), f_2(O), \cdots, f_K(O)$ are K features of the object
  - For simplicity, let's rewrite $c(O)$ as C and $f_i(O)$ as F_i. But it is important to realize that the "randomness" of $c(O), f_i(O)$ is originated from O

$$
\begin{aligned}
p(c|f_1, \cdots, f_K) &= \frac{p(c, f_1, \cdots, f_K)}{p(f_1, \cdots, f_K)} = \frac{p(c)p(f_1, \cdots, f_K|c)}{p(f_1, \cdots, f_K)} && \text{Bayes' rule} \\
&= \frac{p(c)p(f_1|c)\cdots p(f_K|c)}{p(f_1, \cdots, f_K)} && \text{Assume } F_i \perp\!\!\!\perp F_j|C \\
&= \frac{p(c)p(f_1|c)\cdots p(f_K|c)}{p(f_1)\cdots p(f_K)} && \text{If also assume } F_i \perp\!\!\!\perp F_j \\
&= p(c)\frac{p(f_1|c)}{p(f_1)}\cdots\frac{p(f_K|c)}{p(f_K)}
\end{aligned}
$$

A digression: Naive Bayes Algorithm

- In most classification problem, we are interested to compute the most likely class. So we really will go through all possible $c_1, c_2, \cdots$ for $p(c|f_1, \cdots, f_K)$
- Rather than assuming both $F_i \perp\!\!\!\perp F_j | C$ and $F_i \perp\!\!\!\perp F_j$, the latter really is not necessary as we can write

$$p(c|f_1, \cdots, f_K) = \frac{p(c)p(f_1|c)\cdots p(f_K|c)}{\sum_i p(c_i)p(f_1|c_i)\cdots p(f_K|c_i)}$$

  Actually if we only care about which is the most likely class, we can even skip computing the denominator as it is a constant w.r.t. c
- You can find a numerical example here
  - N.B. the author assumes independence of the features in his explanation but the condition is not necessary as noted above

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6. Compute sum/integral

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   - $p(O_1|P_1) = \sum_{i,j} p(G_i|P_1)p(H_j|P_1, G_i)p(O_1|G_i, H_j, P_1)$
     $= p(G_1)(p(H_1|G_1P_1)p(O_1|G_1H_1P_1) + p(H_2|G_1P_1)p(O_1|G_1H_2P_1) + p(H_3|G_1P_1)p(O_1|G_1H_3P_1))$
     $+p(G_2)(p(H_1|G_2P_1)p(O_1|G_2H_1P_1) + p(H_2|G_2P_1)p(O_1|G_2H_2P_1) + p(H_3|G_2P_1)p(O_1|G_2H_3P_1))$
     $+p(G_3)(p(H_1|G_3P_1)p(O_1|G_3H_1P_1) + p(H_2|G_3P_1)p(O_1|G_3H_2P_1) + p(H_3|G_3P_1)p(O_1|G_3H_3P_1))$

Example: Monty Hall problem

Below I will use shorthand like P1, G2, H3 to refer to the case of prize at Door 1, guest picking Door 2, and host open Door 3

1. Introduce helper variables if needed
   - Let's denote O as the other door both guest and host did not pick
2. Identify distributions and condition
   - $P \perp\!\!\!\perp G, O = \{1,2,3\} \setminus \{G, H\}, p(G) = p(H) = \frac{1}{3}$, etc.
3. Identify (conditional) probability to address the question
   - $Pr(Win|switch) = Pr(O = P) = \sum_i p(O_i|P_i)p(P_i) = p(O_1|P_1)$
4. Insert dummy variables to probability by marginalization
   - $p(O_1|P_1) = \sum_{i,j} p(O_1, G_i, H_j|P_1)$
5. Expand probabilities into (conditional) probabilities and evaluate them
   - $p(O_1|P_1) = \sum_{i,j} p(G_i|P_1)p(H_j|P_1, G_i)p(O_1|G_i, H_j, P_1)$
     $= p(G_1)(p(H_1|G_1P_1)p(O_1|G_1H_1P_1) + p(H_2|G_1P_1)p(O_1|G_1H_2P_1) + p(H_3|G_1P_1)p(O_1|G_1H_3P_1))$
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6. Compute sum/integral

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6. Compute sum/integral
   - $p(O_1|P_1) = p(G_2)p(H_3|G_2P_1)p(O_1|G_2H_3P_1) + p(G_3)p(H_2|G_3P_1)p(O_1|G_3H_2P_1) =$
     $\frac{1}{3} \cdot 1 \cdot 1 + \frac{1}{3} \cdot 1 \cdot 1 = \frac{2}{3}$

Epilogue: an engineer (dummy) approach to solve probability problems

Our dummy approach can solve virtually solve any probability problems, but

- Identify what variables to introduced may need some experience
- Can solve any problem with only discrete variables, but if there are too many variables, hand calculation not feasible
  $\Rightarrow$ probabilistic programming
- If continuous variables are involved, the last step may involve intractable integral
  $\Rightarrow$ probabilistic programming

Monte Carlo approach

- Our dummy approach involves some understanding of the problem

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- Of course the computed probability won't be exact
  - Probability estimate improves with # simulations
  - Problem solved as long as we know how to simulate one time (if we don't need exact probability)
  - Even simulation can be hard and computation can be an issue
    $\Rightarrow$ Markov Chain Monte Carlo (MCMC)
    We will delay this to much later

Monte Hall simulation

Algorithm 1 Simulate one game instance

```
P = randint(3)
G = randint(3)
𝓗 = {0, 1, 2} \ {P, G}
if |𝓗| = 2 then
    H = 𝓗[randint(2)]
else
    H = 𝓗[0]
end if
```

More formal treatment: probability space

- More rigorously, a probability model is defined by the **probability space** composed of the triple $(\Omega, \mathcal{F}, p)$
  - Ω is the **sample space** containing all possible outcomes
  - $\mathcal{F}$ is a "σ-field", which is a collection of subsets (events) of Ω
  - p is the (non-negative) **probability measure** on elements of $\mathcal{F}$
- E.g., probability model of unbiased dice
  - $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \cdots, \{1, 2, 3, 4, 5, 6\}\}$
  - $p(S)$ is the probability of an event
    - $p(\{1\}) = p(\{2\}) = p(\{3\}) = p(\{4\}) = p(\{5\}) = p(\{6\}) = 1/6$
    - $p(\{1, 2\}) = p(\{1, 3\}) = \cdots = p(\{5, 6\}) = 2/6$
    - $\cdots$
    - $p(\{1, 2, 3, 4, 5, 6\}) = 1$
- N.B. It could be confusing at first. Be careful that events $\neq$ outcomes. **An event is actually a set of outcomes**

σ-algebra

- The purpose of σ-field (aka σ-algebra) is to impose restriction on what we can and cannot query regarding probability
- Namely, we can only measure the probability of something inside the σ-field $\mathcal{F}$ (i.e., an event)
- Formal definition of σ-field:
  - **σ-field has to satisfied the following: 1) containing empty set $\varnothing$, 2) closed under complement, countable union, and countable intersection of its element**
- E.g., let $\Omega = \{1, 2, 3, 4\}$
  1. $\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is a valid σ-field
  2. $\{\varnothing, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is NOT a valid σ-field
- N.B., A complement, countable union, or countable intersection of Ω is call a **Borel set**
  - $\varnothing, \{1\}, \{1, 2\}$ are example of Borel sets (an event is a Borel set)
  - Collection of all Borel sets forms a σ-algebra (aka Borel (σ-)algebra)

Probability measure

- Probability measure p is a **measure**. Along with $\mathcal{F}$, the tuple $(\mathcal{F}, p)$ forms a **measure space**. For $\mathbb{P}$ to be a valid probability measure, it has to satisfy the following
    - Requirements to be a measure (in the context of measure theory):
        1. $p(\varnothing) = 0$
        2. Countably additive: $p(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} p(A_i), \forall i \neq j, A_i \cap A_j = \varnothing$
    - And since p is a probability measure, it also has to satisfy $p(\Omega) = 1$
- The above constraints are sometimes known as the axioms of probability theory

Some properties of probability measure

From the axioms described in the last slides, one can show that probability measure has to satisfies the following:

1. $p(A^c) = 1 - p(A)$
2. $p(A) \leq p(B)$ if $A \subset B$
3. Union bound: $p(\cup_i A_i) \leq \sum_i p(A_i)$
   - Proof hint: use 2) and induction
4. Inclusion-exclusion formula: $p(\cup_{i=1}^n A_i) = \sum_{i=1}^n p(A_i) - \sum_{i<j} p(A_i \cap A_j) + \sum_{i<j<k} p(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} p(\cap_{i=1}^n A_i)$
   - Proof hint: show $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ and then use induction. ($p(A \cup B) = p(A) + p(B \setminus A)$ and $p(B) = p(A \cap B) + p(B \setminus A)$).

Why so complex?

- Consider X a uniform random variable defined between $[0,1]$
- Define $Y = \begin{cases} 1 & \text{if } X \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$
- Y is a random variable since X is random. It is reasonable to ask what is the probability that $Y=1$. From undergrad probability class,

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dx = ?$$

    - The integral above is actually undefined according to undergrad calculus, where the integral is known as a Riemann integral

- Instead, we have to incorporate the idea of "measure" (Lesbeque integral)

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dp(x) = 0$$

    - The Lesbeque integral above is 0 since the measure of $\{x|x\in[0,1]\cap\mathbb{Q}\}=0$

Some remarks on notation

- In general, we can write

$$p(\Omega') = \int_{\Omega'} dp(\omega)$$

and

$$E[f(X)] = \int_{\Omega} f(X(\omega))dp(\omega)$$

- E.g.,

$$E[X] = \int_{\Omega} X(\omega)dp(\omega) = \int_{\Omega} X(\omega)\ dp = \int_{\Omega} X dp$$

  - Note that p is the probability measure (often people use upper case P instead)
  - People often omit ω as above when context is clear

Lecture 2

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$
- Let's add model type M,
  $p(\theta, o|M) = p(o|M)p(\theta|o, M) = p(\theta|M)p(o|\theta, M)$

Bayes' rule (with model type)

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  $p(\theta, o|M) = p(o|M)p(\theta|o, M) = p(\theta|M)p(o|\theta, M)$

$$\underbrace{p(\theta|o, M)}_{posterior} = \frac{\overbrace{p(\theta|M)}^{prior}\overbrace{p(o|\theta, M)}^{likelihood}}{\underbrace{p(o|M)}_{model\ evidence}}$$

- M: model type
- θ: model parameter
- o: observation

Inference

o: Observed variable, θ: Parameter, x: Latent variable

Maximum Likelihood (ML)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(o|\theta)$

Maximum A Posteriori (MAP)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(\theta|o)$

Bayesian

$$\hat{x} = \sum_x x \underbrace{\sum_\theta p(x|\theta)p(\theta|o)}_{p(x|o)}$$

where $p(\theta|o) = \frac{p(o|\theta)p(\theta)}{p(o)} \propto p(o|\theta)\underbrace{p(\theta)}_{prior}$

Coin Flip

C_1 C_2 C_3



$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

Which coin will I use?

$P(C_1) = 1/3$ $P(C_2) = 1/3$ $P(C_3) = 1/3$

Prior: Probability of a hypothesis before we make any observations

(Slide credit: University of Washington CSE473)

Coin Flip

C_1 C_2 C_3

  

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

Which coin will I use?

$P(C_1) = 1/3$ $P(C_2) = 1/3$ $P(C_3) = 1/3$

Uniform Prior: All hypothesis are equally likely before we make any observations

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin <u>did</u> I use?

$P(C_1|H) = ?$ $P(C_2|H) = ?$ $P(C_3|H) = ?$

$$P(C_1|H) = \frac{\boxed{P(H|C_1)}\boxed{P(C_1)}}{\boxed{P(H)}}$$

$$\boxed{P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)}$$

C_1

$\boxed{P(H|C_1) = 0.1}$

$\boxed{P(C_1) = 1/3}$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.6$

Posterior: Probability of a hypothesis given data

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = ?$ $P(C_2|HT) = ?$ $P(C_3|HT) = ?$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.21$ $P(C_2|HT) = 0.58$ $P(C_3|HT) = 0.21$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin <u>did</u> I use?

$P(C_1|HT) = 0.21$ $P(C_2|HT) = 0.58$ $P(C_3|HT) = 0.21$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:

C_2 

Best estimate for P(H)

$P(H|C_2) = 0.5$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum Likelihood Estimate: The best hypothesis that fits observed data assuming uniform prior

Most likely coin:

C_2 

Best estimate for P(H)

$P(H|C_2) = 0.5$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

- Should we always use Uniform Prior?
- Background knowledge:
  - Heads => you go first in Abalone against TA
  - TAs are nice people
  - => TA is more likely to use a coin biased in your favor

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

We can encode it in the prior:

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin <u>did</u> I use?

$P(C_1|H) = ?$ $P(C_2|H) = ?$ $P(C_3|H) = ?$

$$P(C_1|H) = \alpha P(H|C_1)P(C_1)$$

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

$\boxed{P(C_1) = 0.05 \quad P(C_2) = 0.25 \quad P(C_3) = 0.70}$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin <u>did</u> I use?

$P(C_1|H) = 0.006$ $P(C_2|H) = 0.165$ $P(C_3|H) = 0.829$

ML posterior after Exp 1:

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.600$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 0.05$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 0.25$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

P(C_1|HT) = ?   P(C_2|HT) = ?   P(C_3|HT) = ?

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1	C_2	C_3
		
P(H|C_1) = 0.1	P(H|C_2) = 0.5	P(H|C_3) = 0.9
P(C_1) = 0.05	P(C_2) = 0.25	P(C_3) = 0.70

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1	C_2	C_3
		
$P(H|C_1) = 0.1$	$P(H|C_2) = 0.5$	$P(H|C_3) = 0.9$
$P(C_1) = 0.05$	$P(C_2) = 0.25$	$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:

C_3 

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum A Posteriori (MAP) Estimate: The best hypothesis that fits observed data assuming a non-uniform prior

Most likely coin:

C_3 

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_2 and C_3 are almost equally likely

C_1

C_2

C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

A Better Estimate

Recall: $P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)$ = 0.680

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Bayesian Estimate

Bayesian Estimate: Minimizes prediction error, given data and (generally) assuming a <u>non-uniform prior</u>

$$P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i) = 0.680$$

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Comparison

ML
- Easy to compute

Comparison

ML	• Easy to compute
MAP	• Still relatively easy to compute • Incorporate prior information

Comparison

ML	• Easy to compute
MAP	• Still relatively easy to compute • Incorporate prior information
Bayesian	• Minimizes expected error $\Rightarrow$ especially shines when little data available • Potentially much harder to compute

$$\hat{\mathbf{x}}_{k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k-1}, \mathbf{Q}_k)$$

$$\boxed{\mathbf{F}_k \cdot + \mathbf{B}_k \mathbf{u}_k}$$

(1): $\sim \mathcal{N}(\mathbf{F}_k \hat{\mathbf{x}}_{k-1} + \mathbf{B}_k \mathbf{u}_k, \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^\top)$

$$\tilde{\mathbf{x}}_k$$

$$\boxed{\cdot + \mathbf{w}_k}$$

(2): $\sim \mathcal{N}(\underbrace{\mathbf{F}_k \hat{\mathbf{x}}_{k-1} + \mathbf{B}_k \mathbf{u}_k}_{\hat{\mathbf{x}}_{k|k-1}}, \underbrace{\mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^\top + \mathbf{Q}_k}_{\mathbf{P}_{k|k-1}})$

$$\hat{\mathbf{x}}_k$$

(3): $\sim \mathcal{N}((\mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1} \mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{z}_k, (\mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1})$

$$\boxed{\mathbf{H}_k \cdot}$$

$$\mathbf{y}_k \sim \mathcal{N}(\mathbf{z}_k, \mathbf{R}_k)$$