Information Theory and Probabilistic Programming

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Lecture 1: Overview and review of probabilities

About this course

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   - Understand basic terminology: what is entropy all about?

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2. Statistical inference
   - Bayesian and Monte Carlo techniques
3. Introduction of probabilistic programming
   - Solve inference problems with programming
4. Get better understanding of probability

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  - The theoretical basis of the entire telecom industry is built on top of that
  - Study of extreme cases. What is possible and what is not?

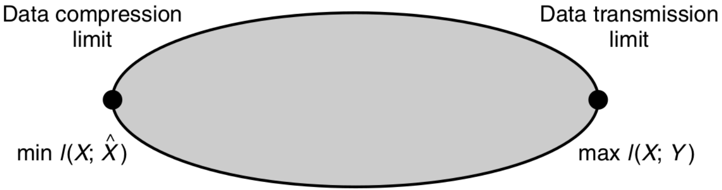

FIGURE 1.2. Information theory as the extreme points of communication theory.

(From Cover and Thomas)

Connection to other fields

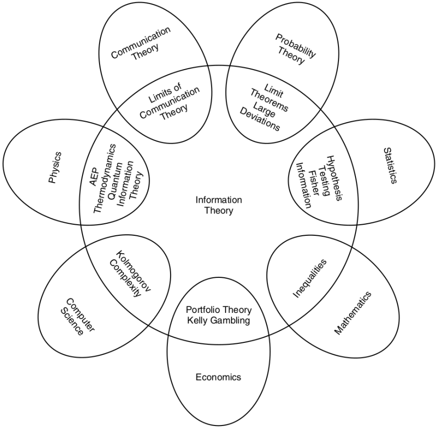

FIGURE 1.1. Relationship of information theory to other fields.

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- Some similar ideas were explored earlier in Bell Labs by Harry Nyquist and Ralph Hartley. But those results are limited to events with equal probability

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A good guess for $H(X=x)$: $\log \frac{1}{p(x)}$

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- We will take the probabilistic view (electrical/communication engineers treatment here) to quantify information theory who usually study with Bayesian models

Neumann-Shannon Anecdote

When Shannon discovered this function he was faced with the need to name it, for it occurred quite often in the theory of communication he was developing. He considered naming it “information” but felt that this word had unfortunate popular interpretations that would interfere with his intended uses of it in the new theory. He was inclined towards naming it “uncertainty” and discussed the matter with the late John Von Neumann. Von Neumann suggested that the function ought to be called “entropy” since it was already in use in some treatises on statistical thermodynamics (e.g. ref. 12). Von Neumann, Shannon reports, suggested that there were two good reasons for calling the function “entropy”. ”It is already in use under that name,” he is reported to have said, ”and besides, it will give you a great edge in debates because nobody really knows what entropy is anyway.” Shannon called the function “entropy” and used it as a measure of “uncertainty,” interchanging the two words in his writings without discrimination.
–From wikipedia

Probability model

- A probability model is used to model uncertain event that can have non-deterministic outcomes
- A probability model can have finite or infinite number of outcomes and even continuous outcomes
- We call the "undetermine" random variable, short for r.v.
- The probability of an **outcome** is the relative chance of getting that outcome
  - For outcome a, we may denote as $Pr(X = a)$ or $p_X(a)$ or even $p(a)$ when it is understood that we are considering variable X
  - $0 \leq p(a) \leq 1$
- We often denote a r.v. using upper case (such as X) and its realization (what was actually observed) using lower case (such as x)

Some probability properties

- Probability mass function (pmf) for discrete random variable (r.v.) X
  - $p(x) \geq 0$
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- Chain rule: $p(x,y,z) = p(x)p(y|x)p(z|x,y)$
  $RHS = p(x)p(y|x)p(z|x,y) = p(x)\frac{p(x,y)}{p(x)}\frac{p(x,y,z)}{p(x,y)} = p(x,y,z) = LHS$

Probabilities and counting

- Six students A, B, C, D, E, F randomly lined up in a row, what is the probability that the order is exactly ABCDEF?
- Six students randomly assigned into two teams (black and white), what is the probability that A,B,C assigned to Team Black and the rest assigned to Team White?

Example: Two jars

- Both Jars A and B have 4 balls
  - Jar A has 1 white and 3 black
  - Jar B has 2 white and 2 black
- Let's draw balls from the jars multiple times. **And put the drawn ball back after each draw.** Can you answer the following?
  - What is the probability of get a white ball from Jar A?
  - What is the probability of getting 3 whites after 6 drawings?
  - If someone randomly pick a jar to draw from and get 3 whites after 6 drawing, what is the probability that he drew from Jar A?

Bayes rule

- Both Jars A and B have 4 balls
  - Jar A has 1 white and 3 black
  - Jar B has 2 white and 2 black
- Say probability of picking Jar A, $Pr(Jar = A) = 0.5$
  - What is the probability of picking from Jar A and getting a white ball $Pr(Jar = A, Ball = white)$?
  - What is $Pr(Ball = white|Jar = A)$?
  - What is $Pr(Jar = A|Ball = white)$?

Expectation

- Recall that $p(x)$ as the distribution of a r.v. X
- The expected value of X is $E[X] \triangleq \sum_x x \cdot p(x)$
- In general, the expected value of a function $f(\cdot)$ of X is $E[f(X)] \triangleq \sum_x f(x) \cdot p(x)$
- Examples
  - $E[X]$ is just the mean of X, often denote as $\overline{X}$
  - The variance of X is $E[(X - \overline{X})^2]$

Independence and conditional independence

- Independence: $p(x,y)=p(x)p(y)$, $X \perp\!\!\!\perp Y$
  - By chain rule, $p(x,y)=p(x)p(y|x)$. Therefore the condition implies that $p(y|x)=p(y)$. In other words, no matter what value X takes, the probability of Y given X is not going to change. So reasonably, they are independent

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- Markov property and conditional independence: $p(x,y|z)=p(x|z)p(y|z)$, $X \perp\!\!\!\perp Y|Z, X \leftrightarrow Z \leftrightarrow Y$
  - Similar to independence, by chain rule, we have $p(x,y|z)=p(x|z)p(y|x,z)$. Along with the above condition, $p(y|x,z)=p(y|z)$. Thus given Z, it does not matter what X supposed to be, the probability of given both variables will not depend on X. Hence, X and Y are conditionally independent given Z
- Caveat: independence and conditional independence are two "independent concepts", we can have both satisfied, none of them satisfied, or one of them satisfied. A common **mistake** is to think that independence leads to conditional independence or vice versa. But that is WRONG

Independence but not conditional independence

Consider flipping two coins with outcomes store as X and Y, say 1 represents a head and 0 represents a tail

- In general the two outcomes should be independent (maybe unless if you are some professional/magical gambler), so we have $X \perp\!\!\!\perp Y$
- Now, let $Z = X \oplus Y$, where $\oplus$ is the exclusive or operation $(1 \oplus 0 = 0 \oplus 1 = 1$ and $1 \oplus 1 = 0 \oplus 0 = 0)$
  - Even though $X \perp\!\!\!\perp Y$, $X \not\perp\!\!\!\perp Y|Z$
  - Actually given Z, X "depends" very much on Y since from $X = Y \oplus Z$, we can find out X precisely given Y
  - We can also check the condition $X \perp\!\!\!\perp Y|Z$ by comparing the probability $p(x|z,y)$ with $p(x|z)$
    - For example, $p_{X|Z}(0|0) = 0.5 \neq 1 = p_{X|Z,Y}(0|0,0)$. Thus $X \perp\!\!\!\perp Y|Z$ cannot be true

A digression: Naive Bayes Algorithm

- Naive Bayes is a simple machine learning algorithm to classify an object with its features
- Basically, we are simply assuming the features are conditionally independent given the object class
- Say if O is the object that $c(O)$ is the corresponding class (can be $c_1, c_2, \cdots$). And say $f_1(O), f_2(O), \cdots, f_K(O)$ are K features of the object
  - For simplicity, let's rewrite $c(O)$ as C and $f_i(O)$ as F_i. But it is important to realize that the "randomness" of $c(O), f_i(O)$ is originated from O

$$
\begin{aligned}
p(c|f_1,\cdots,f_K) &= \frac{p(c,f_1,\cdots,f_K)}{p(f_1,\cdots,f_K)} = \frac{p(c)p(f_1,\cdots,f_K|c)}{p(f_1,\cdots,f_K)} && \text{Bayes' rule} \\
&= \frac{p(c)p(f_1|c)\cdots p(f_K|c)}{p(f_1,\cdots,f_K)} && \text{Assume } F_i \perp\!\!\!\perp F_j|C \\
&= \frac{p(c)p(f_1|c)\cdots p(f_K|c)}{p(f_1)\cdots p(f_K)} && \text{If also assume } F_i \perp\!\!\!\perp F_j \\
&= p(c)\frac{p(f_1|c)}{p(f_1)}\cdots\frac{p(f_K|c)}{p(f_K)}
\end{aligned}
$$

A digression: Naive Bayes Algorithm

- In most classification problem, we are interested to compute the most likely class. So we really will go through all possible $c_1, c_2, \cdots$ for $p(c|f_1, \cdots, f_K)$
- Rather than assuming both $F_i \perp\!\!\!\perp F_j|C$ and $F_i \perp\!\!\!\perp F_j$, the latter really is not necessary as we can write

$$p(c|f_1, \cdots, f_K) = \frac{p(c)p(f_1|c) \cdots p(f_K|c)}{\sum_i p(c_i)p(f_1|c_i) \cdots p(f_K|c_i)}$$

  Actually if we only care about which is the most likely class, we can even skip computing the denominator as it is a constant w.r.t. c
- You can find a numerical example here
  - N.B. the author assumes independence of the features in his explanation but the condition is not necessary as noted above

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6. Compute sum/integral

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   - $p(O_1|P_1) = \sum_{i,j} p(G_i|P_1)p(H_j|P_1, G_i)p(O_1|G_i, H_j, P_1)$
     $= p(G_1)(p(H_1|G_1P_1)p(O_1|G_1H_1P_1) + p(H_2|G_1P_1)p(O_1|G_1H_2P_1) + p(H_3|G_1P_1)p(O_1|G_1H_3P_1))$
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1. Introduce helper variables if needed
   - Let's denote O as the other door both guest and host did not pick
2. Identify distributions and condition
   - $P \perp\!\!\!\perp G, O = \{1,2,3\} \setminus \{G,H\}, p(G) = p(H) = \frac{1}{3}$, etc.
3. Identify (conditional) probability to address the question
   - $Pr(Win|switch) = Pr(O = P) = \sum_i p(O_i|P_i)p(P_i) = p(O_1|P_1)$
4. Insert dummy variables to probability by marginalization
   - $p(O_1|P_1) = \sum_{i,j} p(O_1, G_i, H_j|P_1)$
5. Expand probabilities into (conditional) probabilities and evaluate them
   - $p(O_1|P_1) = \sum_{i,j} p(G_i|P_1)p(H_j|P_1,G_i)p(O_1|G_i,H_j,P_1)$
     $= p(G_1)(p(H_1|G_1P_1)p(O_1|G_1H_1P_1) + p(H_2|G_1P_1)p(O_1|G_1H_2P_1) + p(H_3|G_1P_1)p(O_1|G_1H_3P_1))$
     $+p(G_2)(p(H_1|G_2P_1)p(O_1|G_2H_1P_1) + p(H_2|G_2P_1)p(O_1|G_2H_2P_1) + p(H_3|G_2P_1)p(O_1|G_2H_3P_1))$
     $+p(G_3)(p(H_1|G_3P_1)p(O_1|G_3H_1P_1) + p(H_2|G_3P_1)p(O_1|G_3H_2P_1) + p(H_3|G_3P_1)p(O_1|G_3H_3P_1))$
6. Compute sum/integral

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6. Compute sum/integral
   - $p(O_1|P_1) = p(G_2)p(H_3|G_2P_1)p(O_1|G_2H_3P_1) + p(G_3)p(H_2|G_3P_1)p(O_1|G_3H_2P_1) = \frac{1}{3} \cdot 1 \cdot 1 + \frac{1}{3} \cdot 1 \cdot 1 = \frac{2}{3}$

Epilogue: an engineer (dummy) approach to solve probability problems

Our dummy approach can solve virtually solve any probability problems, but

- Identify what variables to introduced may need some experience
- Can solve any problem with only discrete variables, but if there are too many variables, hand calculation not feasible
  $\Rightarrow$ probabilistic programming
- If continuous variables are involved, the last step may involve intractable integral
  $\Rightarrow$ probabilistic programming

Monte Carlo approach

- Our dummy approach involves some understanding of the problem

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  - Estimate winning probability = # wins / 10,000
- Of course the computed probability won't be exact
  - Probability estimate improves with # simulations
  - Problem solved as long as we know how to simulate one time (if we don't need exact probability)
  - Even simulation can be hard and computation can be an issue
    $\Rightarrow$ Markov Chain Monte Carlo (MCMC)
    We will delay this to much later

Monte Hall simulation

Algorithm 1 Simulate one game instance

$P = randint(3)$
$G = randint(3)$
$\mathcal{H} = \{0, 1, 2\} \setminus \{P, G\}$
if $|\mathcal{H}| = 2$ **then**
    $H = \mathcal{H}[randint(2)]$
else
    $H = \mathcal{H}[0]$
end if

More formal treatment: probability space

- More rigorously, a probability model is defined by the **probability space** composed of the triple $(\Omega, \mathcal{F}, p)$
  - Ω is the **sample space** containing all possible outcomes
  - $\mathcal{F}$ is a "σ-field", which is a collection of subsets (events) of Ω
  - p is the (non-negative) **probability measure** on elements of $\mathcal{F}$
- E.g., probability model of unbiased dice
  - $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \cdots, \{1, 2, 3, 4, 5, 6\}\}$
  - $p(S)$ is the probability of an event
    - $p(\{1\}) = p(\{2\}) = p(\{3\}) = p(\{4\}) = p(\{5\}) = p(\{6\}) = 1/6$
    - $p(\{1, 2\}) = p(\{1, 3\}) = \cdots = p(\{5, 6\}) = 2/6$
    - $\cdots$
    - $p(\{1, 2, 3, 4, 5, 6\}) = 1$
- N.B. It could be confusing at first. Be careful that events $\neq$ outcomes. **An event is actually a set of outcomes**

σ-algebra

- The purpose of σ-field (aka σ-algebra) is to impose restriction on what we can and cannot query regarding probability
- Namely, we can only measure the probability of something inside the σ-field $\mathcal{F}$ (i.e., an event)
- Formal definition of σ-field:
  - **σ-field has to satisfied the following: 1) containing empty set $\varnothing$, 2) closed under complement, countable union, and countable intersection of its element**
- E.g., let $\Omega = \{1, 2, 3, 4\}$
  1. $\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is a valid σ-field
  2. $\{\varnothing, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is NOT a valid σ-field
- N.B., A complement, countable union, or countable intersection of Ω is call a **Borel set**
  - $\varnothing, \{1\}, \{1, 2\}$ are example of Borel sets (an event is a Borel set)
  - Collection of all Borel sets forms a σ-algebra (aka Borel (σ-)algebra)

Probability measure

- Probability measure p is a **measure**. Along with $\mathcal{F}$, the tuple $(\mathcal{F}, p)$ forms a **measure space**. For $\mathbb{P}$ to be a valid probability measure, it has to satisfy the following
  - Requirements to be a measure (in the context of measure theory):
    1. $p(\varnothing) = 0$
    2. Countably additive: $p(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} p(A_i), \forall i \neq j, A_i \cap A_j = \varnothing$
  - And since p is a probability measure, it also has to satisfy $p(\Omega) = 1$
- The above constraints are sometimes known as the axioms of probability theory

Some properties of probability measure

From the axioms described in the last slides, one can show that probability measure has to satisfies the following:

1. $p(A^c) = 1 - p(A)$
2. $p(A) \leq p(B)$ if $A \subset B$
3. Union bound: $p(\cup_i A_i) \leq \sum_i p(A_i)$
   - Proof hint: use 2) and induction
4. Inclusion-exclusion formula:
   $p(\cup_{i=1}^n A_i) = \sum_{i=1}^n p(A_i) - \sum_{i<j} p(A_i \cap A_j) + \sum_{i<j<k} p(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} p(\cap_{i=1}^n A_i)$
   - Proof hint: show $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ and then use induction. ($p(A \cup B) = p(A) + p(B \setminus A)$ and $p(B) = p(A \cap B) + p(B \setminus A)$).

Why so complex?

- Consider X a uniform random variable defined between $[0,1]$
- Define $Y = \begin{cases} 1 & \text{if } X \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$
- Y is a random variable since X is random. It is reasonable to ask what is the probability that $Y = 1$. From undergrad probability class,

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dx = ?$$

  - The integral above is actually undefined according to undergrad calculus, where the integral is known as a Riemann integral
- Instead, we have to incorporate the idea of "measure" (Lesbeque integral)

$$Pr(Y=1) = \int_{\{x|x\in[0,1]\cap\mathbb{Q}\}} dp(x) = 0$$

  - The Lesbeque integral above is 0 since the measure of $\{x|x \in [0,1] \cap \mathbb{Q}\} = 0$

Some remarks on notation

- In general, we can write

$$p(\Omega') = \int_{\Omega'} dp(\omega)$$

and

$$E[f(X)] = \int_{\Omega} f(X(\omega))dp(\omega)$$

- E.g.,

$$E[X] = \int_{\Omega} X(\omega)dp(\omega) = \int_{\Omega} X(\omega)\ dp = \int_{\Omega} X dp$$

  - Note that p is the probability measure (often people use upper case P instead)
  - People often omit ω as above when context is clear

Lecture 2: ML, MAP, and Bayesian estimation

Inference

o: Observed variable, θ: Parameter, x: Latent variable

Maximum Likelihood (ML)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(o|\theta)$

Maximum A Posteriori (MAP)

$\hat{x} = \arg\max_x p(x|\hat{\theta}), \hat{\theta} = \arg\max_\theta p(\theta|o)$

Bayesian

$$\hat{x} = \sum_x x \underbrace{\sum_\theta p(x|\theta)p(\theta|o)}_{p(x|o)}$$

where $p(\theta|o) = \frac{p(o|\theta)p(\theta)}{p(o)} \propto p(o|\theta)\underbrace{p(\theta)}_{prior}$

Coin Flip

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

Which coin will I use?

$P(C_1) = 1/3$ $P(C_2) = 1/3$ $P(C_3) = 1/3$

Prior: Probability of a hypothesis before we make any observations

(Slide credit: University of Washington CSE473)

Coin Flip

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Which coin will I use?

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Uniform Prior: All hypothesis are equally likely before we make any observations

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = ?$ $P(C_2|H) = ?$ $P(C_3|H) = ?$

$$P(C_1|H) = \frac{P(H|C_1)P(C_1)}{P(H)}$$

$$P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.6$

Posterior: Probability of a hypothesis given data

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = ?$ $P(C_2|HT) = ?$ $P(C_3|HT) = ?$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.21$ $P(C_2|HT) = 0.58$ $P(C_3|HT) = 0.21$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 1/3$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 1/3$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = 0.21$ $P(C_2|HT) = 0.58$ $P(C_3|HT) = 0.21$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:	Best estimate for P(H)
C_2 	$P(H\|C_2) = 0.5$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum Likelihood Estimate: The best hypothesis that fits observed data assuming uniform prior

Most likely coin: C_2

Best estimate for P(H)

$P(H|C_2) = 0.5$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 1/3$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

- Should we always use Uniform Prior?
- Background knowledge:
  - Heads => you go first in Abalone against TA
  - TAs are nice people
  - => TA is more likely to use a coin biased in your favor

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Using Prior Knowledge

We can encode it in the prior:

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

C_1 C_2 C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin <u>did</u> I use?

$P(C_1|H) = ?$ $P(C_2|H) = ?$ $P(C_3|H) = ?$

$$P(C_1|H) = \alpha P(H|C_1)P(C_1)$$

C_1	C_2	C_3
		
$P(H|C_1) = 0.1$	$P(H|C_2) = 0.5$	$P(H|C_3) = 0.9$
$P(C_1) = 0.05$	$P(C_2) = 0.25$	$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 1: Heads

Which coin did I use?

$P(C_1|H) = 0.006$ $P(C_2|H) = 0.165$ $P(C_3|H) = 0.829$

ML posterior after Exp 1:

$P(C_1|H) = 0.066$ $P(C_2|H) = 0.333$ $P(C_3|H) = 0.600$

C_1	C_2	C_3
		
$P(H\|C_1) = 0.1$	$P(H\|C_2) = 0.5$	$P(H\|C_3) = 0.9$
$P(C_1) = 0.05$	$P(C_2) = 0.25$	$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin did I use?

$P(C_1|HT) = ?$ $P(C_2|HT) = ?$ $P(C_3|HT) = ?$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1 C_2 C_3

$P(H|C_1) = 0.1$ $P(H|C_2) = 0.5$ $P(H|C_3) = 0.9$

$P(C_1) = 0.05$ $P(C_2) = 0.25$ $P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin <u>did</u> I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

$$P(C_1|HT) = \alpha P(HT|C_1)P(C_1) = \alpha P(H|C_1)P(T|C_1)P(C_1)$$

C_1

$P(H|C_1) = 0.1$

$P(C_1) = 0.05$

C_2

$P(H|C_2) = 0.5$

$P(C_2) = 0.25$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Experiment 2: Tails

Which coin <u>did</u> I use?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

What is the probability of heads after two experiments?

Most likely coin:

C_3

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Your Estimate?

Maximum A Posteriori (MAP) Estimate: The best hypothesis that fits observed data assuming a <u>non-uniform prior</u>

Most likely coin:

C_3 

Best estimate for P(H)

$P(H|C_3) = 0.9$

C_3

$P(H|C_3) = 0.9$

$P(C_3) = 0.70$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Did We Do The Right Thing?

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_2 and C_3 are almost equally likely

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

A Better Estimate

Recall: $P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i)$ = 0.680

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Bayesian Estimate

Bayesian Estimate: Minimizes prediction error, given data and (generally) assuming a <u>non-uniform prior</u>

$$P(H) = \sum_{i=1}^{3} P(H|C_i)P(C_i) = 0.680$$

$P(C_1|HT) = 0.035$ $P(C_2|HT) = 0.481$ $P(C_3|HT) = 0.485$

C_1

$P(H|C_1) = 0.1$

C_2

$P(H|C_2) = 0.5$

C_3

$P(H|C_3) = 0.9$

(Slide credit: University of Washington CSE473)

Comparison

ML
- Easy to compute

Comparison

ML
- Easy to compute

MAP
- Still relatively easy to compute
- Incorporate prior information

Comparison

ML	• Easy to compute
MAP	• Still relatively easy to compute • Incorporate prior information
Bayesian	• Minimizes expected error $\Rightarrow$ especially shines when little data available • Potentially much harder to compute

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$
- Let's add model type M,
  $p(\theta, o|M) = p(o|M)p(\theta|o, M) = p(\theta|M)p(o|\theta, M)$

Bayes' rule (with model type)

- $p(\theta, o) = p(o)p(\theta|o) = p(\theta)p(o|\theta)$
- Let's add model type M,
  $p(\theta, o|M) = p(o|M)p(\theta|o, M) = p(\theta|M)p(o|\theta, M)$

$$\underbrace{p(\theta|o, M)}_{posterior} = \frac{\overbrace{p(\theta|M)}^{prior}\overbrace{p(o|\theta, M)}^{likelihood}}{\underbrace{p(o|M)}_{model\ evidence}}$$

- M: model type
- θ: model parameter
- o: observation

Lecture 3: Common distributions

Gaussian distribution

- By the Central limit theorem, if we add multiple independent variables together, the sum will become more and more like Gaussian
- Gaussian distribution (aka Normal distribution) has a bell shape
  - It is symmetric w.r.t. mean
  - The mean is also the mode
- The pdf is given by

$$\mathcal{N}(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

  where μ is the mean and σ^2 is the variance

Introduction to Multivariate Gaussian

The probability density function (pdf) of a multivariate Gaussian random variable $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

We will also use $\mathcal{N}(\mathbf{x};\boldsymbol{\mu},\Sigma)$ to denote this pdf.

Symmetry and Other Handy Equations

Note that $\mathbf{x}$ and $\boldsymbol{\mu}$ are symmetric in

$$\mathcal{N}(\mathbf{x};\boldsymbol{\mu},\Sigma)=\mathcal{N}(\boldsymbol{\mu};\mathbf{x},\Sigma)=\mathcal{N}(\boldsymbol{\mu}-\mathbf{x};0,\Sigma)=\mathcal{N}(0;\boldsymbol{\mu}-\mathbf{x},\Sigma).$$

These equations are trivial but very handy at times.

Covariance matrix

Σ can be written as $E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top]$

- Eigenvalues are the variance along the principal axes (directions where variable changes the most)
    - $\therefore$ eigenvalues are real and ≥ 0 in general
    - If we don't assume the degenerate case where the vector variables do not vary in some directions, then all eigenvalues $> 0 \Rightarrow \Sigma^{-1}$ exists

Marginalization of normal distribution

- Consider $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and let say $\mathbf{X}$ is a segment of $\mathbf{Z}$. That is, $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$ for some $\mathbf{Y}$. Then how should $\mathbf{X}$ behave?

Marginalization of normal distribution

- Consider $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and let say $\mathbf{X}$ is a segment of $\mathbf{Z}$. That is, $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$ for some $\mathbf{Y}$. Then how should $\mathbf{X}$ behave?
- We can find the pdf of $\mathbf{X}$ by just marginalizing that of $\mathbf{Z}$. That is

$$
\begin{aligned}
p(\mathbf{x}) &= \int p(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
&= \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int \exp\left(-\frac{1}{2} \begin{pmatrix} \mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} \mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix} \right) d\mathbf{y}
\end{aligned}
$$

Marginalization of normal distribution

- Denote Σ^{-1} as Λ (also known as the precision matrix). And partition both Σ and Λ into $\Sigma = \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \Lambda_{\mathbf{XX}} & \Lambda_{\mathbf{XY}} \\ \Lambda_{\mathbf{YX}} & \Lambda_{\mathbf{YY}} \end{pmatrix}$

Marginalization of normal distribution

- Denote Σ^{-1} as Λ (also known as the precision matrix). And partition both Σ and Λ into $\Sigma = \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \Lambda_{\mathbf{XX}} & \Lambda_{\mathbf{XY}} \\ \Lambda_{\mathbf{YX}} & \Lambda_{\mathbf{YY}} \end{pmatrix}$
- Then we have

$$
\begin{aligned}
p(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int \exp\Bigg(-\frac{1}{2} \big[(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}) \\
+ (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}) + (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XY}} (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}) \\
+ (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YY}} (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}) \big] \Bigg) d\mathbf{y} \\
= \frac{e^{-\frac{(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XX}} (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int \exp\Bigg(-\frac{1}{2} \big[(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}) \\
+ (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})^T \Lambda_{\mathbf{XY}} (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}) + (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})^T \Lambda_{\mathbf{YY}} (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}) \big] \Bigg) d\mathbf{y}
\end{aligned}
$$

Marginalization of normal distribution

To proceed, let's apply the completing square trick on $(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YX}}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})+(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T\Lambda_{\mathbf{XY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})+(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})$. For the ease of exposition, let us denote $\tilde{\mathbf{x}}$ as $\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}$ and $\tilde{\mathbf{y}}$ as $\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}$. We have

Marginalization of normal distribution

To proceed, let's apply the completing square trick on $(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YX}}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})+(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T\Lambda_{\mathbf{XY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})+(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})^T\Lambda_{\mathbf{YY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}})$. For the ease of exposition, let us denote $\tilde{\mathbf{x}}$ as $\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}$ and $\tilde{\mathbf{y}}$ as $\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}$. We have

$$
\begin{aligned}
&\tilde{\mathbf{y}}^T\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}}+\tilde{\mathbf{x}}^T\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}}+\tilde{\mathbf{y}}^T\Lambda_{\mathbf{YY}}\tilde{\mathbf{y}}\\
=&(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})-\tilde{\mathbf{x}}^T\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}},
\end{aligned}
$$

where we use the fact that $\Lambda=\Sigma^{-1}$ is symmetric and so $\Lambda_{\mathbf{XY}}=\Lambda_{\mathbf{YX}}$

Marginalization of normal distribution

$$p(\mathbf{x}) = \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}} d\mathbf{y}$$

Marginalization of normal distribution

$$p(\mathbf{x}) = \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}}\int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}}d\mathbf{y}$$

$$= \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}\right)$$

Marginalization of normal distribution

$$
\begin{aligned}
p(\mathbf{x}) &= \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}}\int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}}d\mathbf{y} \\
&= \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}\right) \\
&\overset{(a)}{=} \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right)
\end{aligned}
$$

Marginalization of normal distribution

$$
\begin{aligned}
p(\mathbf{x}) &= \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}} d\mathbf{y} \\
&= \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}\right) \\
&\overset{(a)}{=} \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right) \\
&\overset{(b)}{=} \frac{1}{\sqrt{\det(2\pi\Sigma_{\mathbf{XX}})}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right)
\end{aligned}
$$

Marginalization of normal distribution

$$
\begin{aligned}
p(\mathbf{x}) &= \frac{e^{-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}}}{\sqrt{\det(2\pi\Sigma)}} \int e^{-\frac{(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})^T\Lambda_{\mathbf{YY}}(\tilde{\mathbf{y}}+\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}\tilde{\mathbf{x}})}{2}} d\mathbf{y} \\
&= \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T(\Lambda_{\mathbf{XX}}-\Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}})\tilde{\mathbf{x}}}{2}\right) \\
&\overset{(a)}{=} \frac{\sqrt{\det(2\pi\Lambda_{\mathbf{YY}}^{-1})}}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right) \\
&\overset{(b)}{=} \frac{1}{\sqrt{\det(2\pi\Sigma_{\mathbf{XX}})}} \exp\left(-\frac{\tilde{\mathbf{x}}^T\Sigma_{\mathbf{XX}}^{-1}\tilde{\mathbf{x}}}{2}\right) \\
&= \frac{1}{\sqrt{\det(2\pi\Sigma_{\mathbf{XX}})}} \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T\Sigma_{\mathbf{XX}}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})}{2}\right),
\end{aligned}
$$

where (a) and (b) will be shown next

(a) $\Sigma_{\mathbf{XX}}^{-1} = \Lambda_{\mathbf{XX}} - \Lambda_{\mathbf{XY}}\Lambda_{\mathbf{YY}}^{-1}\Lambda_{\mathbf{YX}}$

Lemma

Assume $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$, *then* $A^{-1} = \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$

Proof.

Note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Thus $A\tilde{A} + B\tilde{C} = I$ and $A\tilde{B} + B\tilde{D} = 0$. So

$A(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}) = A\tilde{A} - (A\tilde{B})\tilde{D}^{-1}\tilde{C} = A\tilde{A} + B\tilde{D}\tilde{D}^{-1}\tilde{C} = A\tilde{A} + B\tilde{C} = I$ □

(b) $\det(a\Sigma) = \det(a\Sigma_{\mathbf{YY}})\det(a\Lambda_{\mathbf{XX}}^{-1})$

Lemma

Assume $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$, *then* $det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(D)det(\tilde{A}^{-1})$

Proof.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A & B \\ D^{-1}C & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}$$

$$\Rightarrow det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(D)det(A - BD^{-1}C) = det(D)det(\tilde{A}^{-1})$$

□

Remark

N.B. $A - BD^{-1}C$ *is known as Schur complement*

Conditioning mulitivariate Gaussian

- Consider the same $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$. What will $\mathbf{X}$ be like if $\mathbf{Y}$ is observed to be $\mathbf{y}$?

Conditioning mulitivariate Gaussian

- Consider the same $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$. What will $\mathbf{X}$ be like if $\mathbf{Y}$ is observed to be $\mathbf{y}$?
- Basically, we want to find $p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}, \mathbf{y})/p(\mathbf{y})$

Conditioning mulitivariate Gaussian

- Consider the same $\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$ and $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$. What will $\mathbf{X}$ be like if $\mathbf{Y}$ is observed to be $\mathbf{y}$?
- Basically, we want to find $p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}, \mathbf{y})/p(\mathbf{y})$
- From previous result, we have $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}; \boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}})$. Therefore,

$$\begin{aligned} p(\mathbf{x}|\mathbf{y}) &\propto \exp\left(-\frac{1}{2}\left[\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{pmatrix} - \tilde{\mathbf{y}}^T \Sigma_{\mathbf{YY}}^{-1} \tilde{\mathbf{y}}\right]\right) \\ &\propto \exp\left(-\frac{1}{2}[\tilde{\mathbf{x}}^T \Lambda_{\mathbf{XX}} \tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T \Lambda_{\mathbf{XY}} \tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T \Lambda_{\mathbf{YX}} \tilde{\mathbf{x}}]\right), \end{aligned}$$

where we use $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ as shorthands of $\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}$ and $\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}$ as before

Conditioning mulitivariate Gaussian

- Completing the square for $\tilde{\mathbf{x}}$, we have

$$
\begin{aligned}
p(\mathbf{x}|\mathbf{y}) &\propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}}+\Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})^T\Lambda_{\mathbf{XX}}(\tilde{\mathbf{x}}+\Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})\right)\\
&= \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}+\Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}))^T\Lambda_{\mathbf{XX}}\right.\\
&\qquad\left.(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}+\Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{Y}}))\right)
\end{aligned}
$$

Conditioning mulitivariate Gaussian

- Completing the square for $\tilde{\mathbf{x}}$, we have

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})^T\Lambda_{\mathbf{XX}}(\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})\right)$$

$$= \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))^T\Lambda_{\mathbf{XX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))\right)$$

- Therefore $\mathbf{X}|\mathbf{y}$ is Gaussian distributed with mean $\boldsymbol{\mu}_{\mathbf{X}} - \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})$ and covariance $\Lambda_{\mathbf{XX}}^{-1}$

Conditioning mulitivariate Gaussian

- Completing the square for $\tilde{\mathbf{x}}$, we have

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}(\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})^T \Lambda_{\mathbf{XX}} (\tilde{\mathbf{x}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}\tilde{\mathbf{y}})\right)$$
$$= \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))^T \Lambda_{\mathbf{XX}} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}} + \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}))\right)$$

- Therefore $\mathbf{X}|\mathbf{y}$ is Gaussian distributed with mean $\boldsymbol{\mu}_{\mathbf{X}} - \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})$ and covariance $\Lambda_{\mathbf{XX}}^{-1}$
- Note that since $\Lambda_{\mathbf{XX}}\Sigma_{\mathbf{XY}} + \Lambda_{\mathbf{XY}}\Sigma_{\mathbf{YY}} = 0 \Rightarrow \Lambda_{\mathbf{XX}}^{-1}\Lambda_{\mathbf{XY}} = -\Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}$ and from (a), we have

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}}),$$

where $\Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}} \triangleq \Sigma|\Sigma_{\mathbf{YY}}$ is a Schur complement

Conditioning mulitivariate Gaussian

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

- When the observation of $\mathbf{Y}$ is exactly the mean, the conditioned mean does not change

Conditioning mulitivariate Gaussian

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

- When the observation of $\mathbf{Y}$ is exactly the mean, the conditioned mean does not change
- Otherwise, it needs to be modified and the size of the adjustment decreases with $\Sigma_{\mathbf{YY}}$, the variance of $\mathbf{Y}$ for the 1-D case.
  - The observation is less reliable with the increase of $\Sigma_{\mathbf{YY}}$. The adjustment is finally scaled by $\Sigma_{\mathbf{XY}}$, which translates the variation of $\mathbf{Y}$ to the variation of $\mathbf{X}$

Conditioning mulitivariate Gaussian

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

- When the observation of $\mathbf{Y}$ is exactly the mean, the conditioned mean does not change
- Otherwise, it needs to be modified and the size of the adjustment decreases with $\Sigma_{\mathbf{YY}}$, the variance of $\mathbf{Y}$ for the 1-D case.
  - The observation is less reliable with the increase of $\Sigma_{\mathbf{YY}}$. The adjustment is finally scaled by $\Sigma_{\mathbf{XY}}$, which translates the variation of $\mathbf{Y}$ to the variation of $\mathbf{X}$
  - In particular, if $\mathbf{X}$ and $\mathbf{Y}$ are negatively correlated, the sign of the adjustment will be reversed

Conditioning mulitivariate Gaussian

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

- When the observation of $\mathbf{Y}$ is exactly the mean, the conditioned mean does not change
- Otherwise, it needs to be modified and the size of the adjustment decreases with $\Sigma_{\mathbf{YY}}$, the variance of $\mathbf{Y}$ for the 1-D case.
  - The observation is less reliable with the increase of $\Sigma_{\mathbf{YY}}$. The adjustment is finally scaled by $\Sigma_{\mathbf{XY}}$, which translates the variation of $\mathbf{Y}$ to the variation of $\mathbf{X}$
  - In particular, if $\mathbf{X}$ and $\mathbf{Y}$ are negatively correlated, the sign of the adjustment will be reversed
- As for the variance of the conditioned variable, it always decreases and the decrease is larger if $\Sigma_{\mathbf{YY}}$ is smaller and $\Sigma_{\mathbf{XY}}$ is larger ($\mathbf{X}$ and $\mathbf{Y}$ are more correlated)

What is a Gaussian Process?

A Gaussian Process (GP) is a collection of random variables, any finite number of which have a joint Gaussian distribution.

$$f(x) \sim \mathsf{GP}(m(x), k(x, x'))$$

- $f(x)$ is the function to be modeled.
- $m(x)$ is the mean function, usually zero.
- $k(x, x')$ is the covariance function or kernel.

Advantages and Disadvantages

Advantages:

- Flexible
- Probabilistic Nature
- Non-Parametric

Disadvantages:

- Computational Complexity
- Hyperparameter Sensitivity

Applications

- Regression and function estimation
- Time series forecasting
- Optimization

Uncorrelated implies independence

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}}), \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}})$$

If $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated, $\Sigma_{\mathbf{XY}} = 0$. Then

$$\mathbf{X}|\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}}, \Sigma_{\mathbf{XX}})$$

Note that the statistics of $\mathbf{X}$ does not change with respect to $\mathbf{y}$ and so $\mathbf{X}$ is independent of $\mathbf{Y}$

$X \perp Y | Z$ if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$

Corollary

Given multivariate Gaussian variables X, Y and Z, we have X and Y are conditionally independent given Z if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$, where $\rho_{XZ} = \frac{E[(X-E(X))(Z-E(Z))]}{\sqrt{E[(X-E(X))^2]E[(Z-E(Z))^2]}}$ is the correlation coefficent between X and Z. Similarly, ρ_{YZ} and ρ_{XY} are the correlation coefficients between Y and Z, and X and Y, respectively.

$X \perp Y | Z$ if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$

Proof.

- From the definition of correlation coefficient, $\Sigma = \begin{pmatrix} \sigma_{XX} & \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} & \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} \\ \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} & \sigma_{YY} & \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} \\ \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} & \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} & \sigma_{ZZ} \end{pmatrix}$

$X \perp\!\!\!\perp Y|Z$ if $\rho_{XZ}\rho_{YZ} = \rho_{XY}$

Proof.

- From the definition of correlation coefficient, $\Sigma = \begin{pmatrix} \sigma_{XX} & \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} & \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} \\ \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} & \sigma_{YY} & \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} \\ \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} & \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} & \sigma_{ZZ} \end{pmatrix}$
- Then from the conditioning result, we have

$$\Sigma_{\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)|Z} = \begin{pmatrix} \sigma_{XX} & \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} \\ \sqrt{\sigma_{XX}\sigma_{YY}}\rho_{XY} & \sigma_{YY} \end{pmatrix} - \begin{pmatrix} \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} & \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} \end{pmatrix} \sigma_{ZZ}^{-1} \begin{pmatrix} \sqrt{\sigma_{XX}\sigma_{ZZ}}\rho_{XZ} \\ \sqrt{\sigma_{YY}\sigma_{ZZ}}\rho_{YZ} \end{pmatrix}$$

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- Therefore, X and Y are uncorrelated given Z when the off-diagonal is zero and this gives us $\rho_{XY} = \rho_{XZ}\rho_{YZ}$. Since for Gaussian variables, uncorrelatedness implies independence. This concludes the proof.

□

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- The mean and variance are

$$E[X] = p \cdot 1 + (1-p) \cdot 0 = p$$

$$Var[X] = p \cdot (1-p)^2 + (1-p) \cdot p^2 = p(1-p)$$

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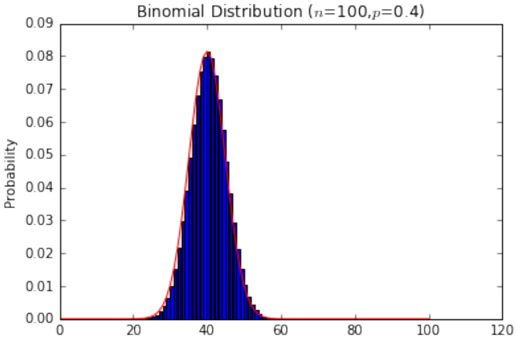
    - Therefore,
    $Var[X] = E[X^2] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2 = N(N-1)p^2 + Np - (Np)^2 = Np(1-p)$

Binomial distribution

As shown below, the binomial distribution can be model well with a normal distribution $\mathcal{N}(Np, Np(1-p))$ for large N



The binomial distribution is shown in blue and an approximation by normal distribution is shown in red

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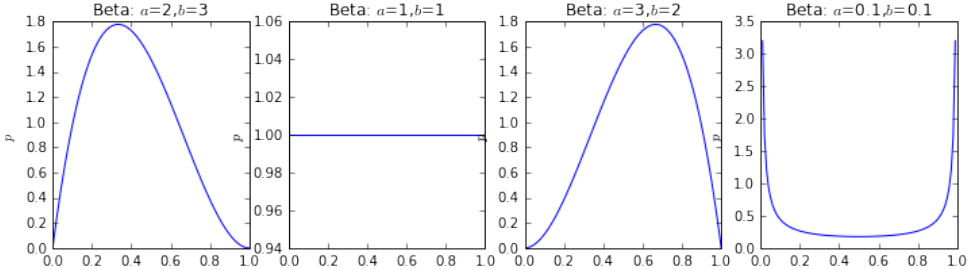
- However, if we select $p(p)$ of a form $p(p) \propto p^a(1-p)^b$, then the resulting posterior distribution with the same form as before. This choice is often chosen for practical purposes, and a prior with same "form" as its likelihood (and thus posterior) is known as the conjugate prior

Beta distribution

- The conjugate prior of both Bernoulli and binomial distributions is the beta distribution. Its pdf is given by

$$Beta(x|a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)},$$

where $X \in [0,1]$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$



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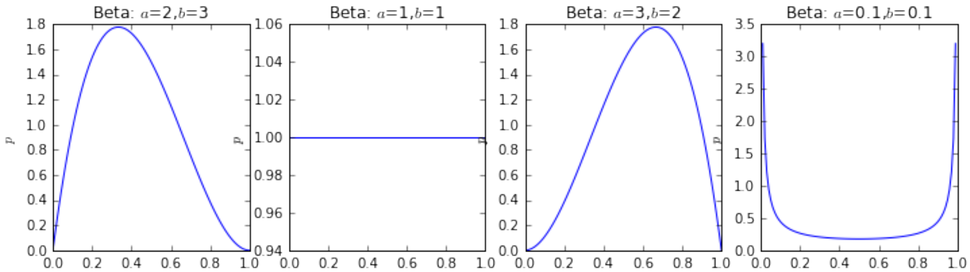

- Note that with $a=b=1$, $Beta(x|1,1)=1$. It is the same as no prior

Gamma function

Note that $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx$

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- Therefore, for integer $z > 1$, $\Gamma(z) = (z-1)!$

Mode of beta distribution

- The mode is the peak of a distribution. Recall that $Beta(x|a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$. Set

$$\frac{\partial Beta(x|a,b)}{\partial x} = \frac{(a-1)x^{a-2}(1-x)^{b-1} - (b-1)x^{a-1}(1-x)^{b-2}}{B(a,b)} = 0,$$

  we have $(a-1)(1-x) = (b-1)x \Rightarrow x = \frac{a-1}{a+b-2}$ when $a, b > 1$
- Note that when a or b is less than or equal to 1, the peak appears at either $x = 0$ or $x = 1$

Mean and variance of Beta distribution

Note that $\int_{x=0}^{1} p(x|a,b) = 1 \Rightarrow \int_{x=0}^{1} x^{a-1}(1-x)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. This gives us a handy trick to manipulate beta distribution

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\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2} \\
&= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} = \frac{ab}{(a+b)^2(a+b+1)}
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Posterior estimate of probability p

Consider the coin flipping example again. Let say the prior probability[1] of the coin is beta distributed with parameters a and b. And we flip the coin once to get outcome x.

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$$\begin{aligned} p(p|x,a,b) =& Const1 \cdot Beta(p|a,b) Bern(x|p) \\ =& Const2 \cdot p^{a-1+x}(1-p)^{b-1+1-x} \\ =& Beta(p|\tilde{a},\tilde{b}) \end{aligned}$$

So the posterior probability distribution is also beta distributed and the parameters just changed to $\tilde{a} \leftarrow a + x$ and $\tilde{b} \leftarrow b + 1 - x$

[1]Note that this can be very confusing at the beginning. Beware that we are talking about the distribution of the probability of some outcome

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Let say we continue our example and we flip the coin by N times and obtain x head. So instead of the Bernoulli likelihood, we have a binomial likelihood. Like the last slide, we have the same beta prior with parameters a and b.

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\begin{aligned}
p(p|x,a,b) =& Const1 \cdot Beta(p|a,b) Bin(x|p,N) \\
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Again, the posterior distribution is still beta but with parameters updated to $\tilde{a} \leftarrow a + x$ and $\tilde{b} \leftarrow b + N - x$

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- Another coin example
  - Fall back to high school, assume that we flip a coin for 10 times and got 3 heads. We want to estimate the chance of getting heads

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  - 0? Okay, the estimate is a bit extreme. We know that it is very difficult to make a coin that always gives a tail
  - How about we first assumed that we actually flipped two times and got 1 head before we did experiment? We will estimate 1/12 instead of 0/10

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- Recall that $Beta(1,1)=1$ and so likelihood function is equivalent to $Beta(p|1,1)Bin(0|p,10) \sim Beta(1,11)$. Thus the ML estimate is the mode of $Beta(1,11) \Rightarrow p_{Head}^{(ML)} = \frac{1-1}{1+11-2} = \frac{0}{10} = 0$
  - This indeed is the same as our high school naïve estimate

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- Note that Bayesian estimation is "self-regularized" (i.e., giving less extreme results) since it inherently averages out all possible cases

Remark

Note that we used the non-informative prior above just to illustrate the self-regularization property of Bayesian estimation. When you are given a prior, you should always use the given prior instead for an actual problem

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- Just make sure we are in the same pace. Note that $p_1 + p_2 + \cdots + p_n = 1$ and $x_1 + x_2 + \cdots + x_n = N$

Dirichlet distribution

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- It turns out that the distribution is the so-called Dirichlet distribution. Its pdf is given by

$$
\begin{aligned}
&Dir(x_1,\cdots,x_n|\alpha_1,\cdots,\alpha_n)\\
=&\frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}
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- As usual since pdf should be normalized to 1, we have

$$\int x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}$$

Mean, mode, variance of Dirichlet distribution

- Mean:

$$E[X_1] = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \int x_1^{\alpha_1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1}$$

$$= \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \frac{\Gamma(\alpha_1 + 1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)} = \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n}$$

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  $Var(X_1) = E[X_1^2] - E[X_1^2] = \frac{(\alpha_1+1)\alpha_1}{(\alpha_1+\cdots+\alpha_n+1)(\alpha_1+\cdots+\alpha_n)} - \frac{\alpha_1^2}{(\alpha_1+\cdots+\alpha_n)^2} = \frac{\alpha_1(\alpha_0-\alpha_1)}{\alpha_0^2(\alpha_0+1)}$, where $\alpha_0 = \alpha_1+\cdots+\alpha_n$
- Mode: one can show that the mode of $Dir(\alpha_1,\cdots,\alpha_n)$ for $\alpha_1,\cdots,\alpha_n > 1$ is

$$\frac{\alpha_i - 1}{\alpha_1+\cdots+\alpha_n - n}.$$

We will not show it now but will leave as an exercise

Summary of Dirichlet distribution

- Pdf:

$$Dir(\mathbf{x}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1}$$

- Mean:

$$\frac{\alpha_i}{\alpha_1 + \cdots + \alpha_n}$$

- Variance:

$$\frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$$

- Mode:

$$\frac{\alpha_i - 1}{\alpha_1 + \cdots + \alpha_n - n}$$

Posterior probability given Multinomial likelihood and Dirichlet prior

Upon observing $x_1, \cdots, x_n$, the posterior distribution of $p_1, \cdots, p_n$ becomes

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=& Dir(p_1, \cdots, p_n | \tilde{\alpha}_1, \cdots, \tilde{\alpha}_n)
\end{aligned}$$

So the posterior distribution is Dirichlet with parameters updated to $\tilde{\alpha}_1 \leftarrow x_1 + \alpha_1, \cdots, \tilde{\alpha}_n \leftarrow x_n + \alpha_n$

Poisson distribution

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where k is a non-negative integer, λ is rate of arrival and T is the length of the observed period. It is easy to check that (please verify)

$$Mean = \lambda T$$

$$Variance = \lambda T$$

N.B. the parameters λT comes as a group and so we can consider it as a single parameter

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   - It can be less perfect to model the times my car broke down. The events are likely to be related

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  $\approx \frac{N^k}{k!}\lambda^k\frac{T^k}{N^k}(1-\lambda\Delta)^{N-k}$

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Note that indeed $Pr(k \text{ arrivals in } T) = Poisson(k|\lambda T)$

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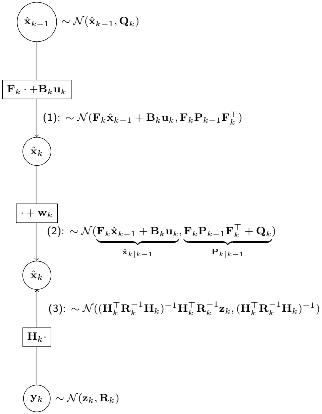
Exponential distribution

$f_T(t) = \lambda\exp(-\lambda t) \triangleq Exp(t|\lambda)$ is the pdf of the exponential distribution with parameter λ. It is easy to verify that (as exercise)

- $E[T] = 1/\lambda$
- $Var(T) = 1/\lambda^2$


$\hat{\mathbf{x}}_{k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k-1}, \mathbf{Q}_k)$
$\mathbf{F}_k \cdot + \mathbf{B}_k \mathbf{u}_k$
(1): $\sim \mathcal{N}(\mathbf{F}_k \hat{\mathbf{x}}_{k-1} + \mathbf{B}_k \mathbf{u}_k, \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^\top)$
$\tilde{\mathbf{x}}_k$
$\cdot + \mathbf{w}_k$
(2): $\sim \mathcal{N}(\underbrace{\mathbf{F}_k \hat{\mathbf{x}}_{k-1} + \mathbf{B}_k \mathbf{u}_k}_{\hat{\mathbf{x}}_{k|k-1}}, \underbrace{\mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^\top + \mathbf{Q}_k}_{\mathbf{P}_{k|k-1}})$
$\hat{\mathbf{x}}_k$
(3): $\sim \mathcal{N}((\mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1} \mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{z}_k, (\mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1})$
$\mathbf{H}_k \cdot$
$\mathbf{y}_k \sim \mathcal{N}(\mathbf{z}_k, \mathbf{R}_k)$